Quaternion-valued KdV Solutions

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Abstract: The KdV equation is a prototypical example of a completely solvable nonlinear partial differential equation due in part to its admittance of soliton solutions. In recent years, there has been a growing interest in exploring the non-commutative KdV equation. In this paper, the non-commutative KdV equation is derived from the nonabelian KP hierarchy and solved through a generalized Darboux transformation. Three different solution types are explored with detailed descriptions of 1-soliton and 2-soliton properties and interactions. Finally, 2-soliton scattering is analyzed with respect to the parameters of the involved 1-solitons.

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1 Introduction and Background

1.1 Introduction

The Korteweg-de Vries (KdV) equation,

$$u_t = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}, \tag{1}$$

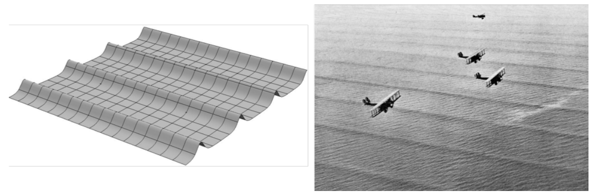

Figure 1: Graphic demonstrating how the KP equation, a closely related soliton equation, models ocean surface waves.

remains the canonical example of a completely integrable nonlinear partial differential equation. Although originally derived by Boussinesq, the famous KdV equation resurfaced in a famous 19th-century paper of Korteweg and de Vries [1] to model surface waves in a canal. The KdV equation has subsequently been found to be fundamental in the description of a wide array of physical phenomena including plasma physics, electrical transmission, and even Jupiter's Great Red Spot [2]. More importantly, the theory developed for the integrability of the KdV equation represents one of the most far-reaching breakthroughs in nonlinear mathematics. The resulting mathematical theory connected by the KdV equation is interesting and vast, uniting results from algebraic geometry to quantum field theory. Much of this fame comes from the KdV equation being the first equation known to admit *soliton* solutions.

Localized solutions to dispersive equations typically dissipate as $t \to \infty$, as might be expected of the water waves resulting from a rock thrown in a pond. However, one does see water waves in nature that move at a constant speed without changing shape for an astoundingly long periods of time. These *solitary waves* or solitons were first observed by John Scott Russell in 1834, who followed a well-defined heap of water down a Scottish canal on horseback for over a mile. Boussinesq first wrote down a family of 1-soliton solutions

$$u_1(x,t) = u_1(x,t;k,\xi) = 2k^2 \operatorname{sech}^2(\eta(x,t;k,\xi)) \tag{2}$$

$$\eta(x,t;k,\xi) = kx + k^3 t + \xi \tag{3}$$

depending on the choice of parameters k and ξ. This solution, graphed for each time t, produces a translating solitary wave like Russell's traveling to the left at speed k^2 with height $2k^2$.

Incredibly, just as one can combine solutions to linear ordinary differential equations, there is a sense of *nonlinear* superposition between individual 1-soliton solutions that allows for the construction of *multisoliton* solutions or n-soliton solutions that appear to be composed of many superimposed 1-solitons. However, multisoliton solutions deviate from a simple superposition of 1-solitons since the solitons affect the trajectories of other solitons upon collision. In particular, the faster solitons seem to shift forwards slightly, and the slower ones shift backwards – these shifts are called the *phase shifts*. As shown in Figure 1.1, phase shifts describe particle-like collisions in which colliding solitons appear to transfer their amplitudes to one another. In the real case, the phase shifts are wholly dependent on the incoming velocities of the involved 1-solitons. More about the phase shifts in the real case can be seen in [3].

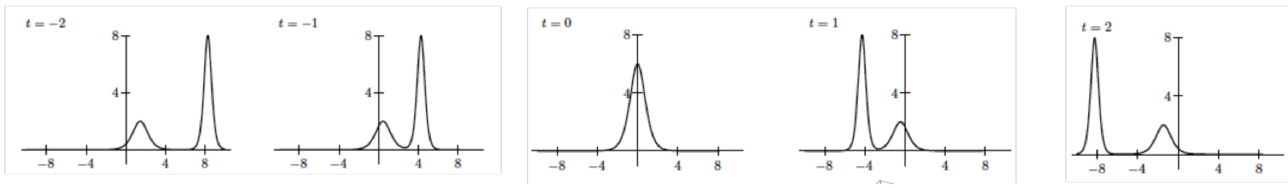

Figure 2: Graphic depicting a real 2-soliton solution to the KdV equation at different times. Looking at $t = -1$ and $t = 1$ shows the famed *phase shift*.

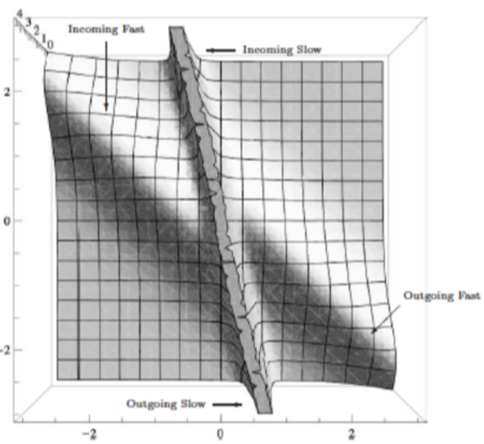

Figure 3: A 3D plot of the 2-soliton interaction from Figure 1.1 demonstrating the phase shift occuring on impact at $t = 0$. In this plot, each vertical slice represents a frame of the time dependent soliton $u(x, t)$.

Although interactions such as these are well known in the real and complex cases, they are unexplored over the quaternions. Aside from presenting an appealing mathematical challenge, soliton theory often finds applications in areas where noncommutativity plays an important role; most notably in quantum mechanics. Exploration of the KdV equation over the quaternions is particularly appealing as the quaternions are the simplest non-commutative associative algebra and the unique non-commutative associative division algebra.

The paper proceeds by obtaining solutions to non-commutative KdV equation through a generalization of the *Darboux transformation*[1], an early technique used to solve the (commutative) KdV equation[2] Then, an alternative iterative scheme involving quasi-determinants is constructed for efficient computation through Mathematica. Periodic solutions and rational solutions are explored briefly, followed by an examination of 1-soliton and 2-soliton solutions including a description of their interactions. Using a theorem linking solution "shifts" to the application of constant coefficient differential operators to KdV-Darboux kernels, the direct scattering problem is posed and solved.

This paper seeks to introduce the necessary concepts to understand the KdV equation, the role of solitons in the solution-finding process, and the subsequent generalization over the quaternions from the starting point of an interested undergraduate mathematics major. For this reason, many expository sections have been added to introduce quaternions, differential operators, and to convey some of the intuition behind otherwise "magical" proof techniques introduced for a result such as the Darboux transformation. Much of the exposition is influenced heavily by, and stated more clearly in, the undergraduate presentation of soliton theory in [4]. Sections or Theorems that is solely original work by the author and advisor Dr. Alex Kasman are marked by ★. An ever growing appendix section has been included to convey interesting but logically independent topics in soliton theory, including the KP hierarchy, the Bilinear KdV equation and τ-functions, and an algebrogeometric object called the Grassmann Cone $\Gamma_{2,4}$

[1] or can be thought to be formulated from nonabelian KP hierarchy, more in the Appendix.

[2] The Darboux transformation has been more recently subsumed into the Bäcklund transformation in 1883 by Bäcklund in [5].

spanned by the solution set to the bilinear KP equation. It is my hope that this presentation will make my research under Dr. Kasman during the summer of 2018 and my senior year approachable to myself in later years, aid those wishing for a foothold into soliton theory, and convey some of the interesting connections responsible for the high variety of perspectives with which to view modern soliton theory.

1.2 Quaternions

Throughout the paper, let

$$\mathbb{H} = \{q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

be the four-dimensional *quaternion algebra* over $\mathbb{R}$ with the multiplication rules

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

$\mathbb{H}$ is an associative non-commutative division ring invented by W. R. Hamilton in 1843. Similar to Russell's discovery of the wave of translation, Hamilton discovered the quaternions next to an Irish canal. As the story goes, Hamilton was fruitlessly attempting to extend complex operations into three dimensions[3] when he reached an epiphany leading to the development of the quaternions while crossing the Broome Bridge of the Royal Canal. He then famous carved the equations above into the bridge, which remains there today.

Whenever $q \in \mathbb{H}$ is a quaternion, then $q_i \in \mathbb{R}$ for $0 \leq i \leq 3$ will denote its components according to the convention $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$. In some cases, the alternative notation $q = q_0 + \vec{q}$ will be used, where q_0 is the *real* part and $\vec{q} = q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$ is the *vector* or *pure* part. Accordingly, quaternions with real part zero are referred to as pure quaternions. The appeal of this notation will be apparent shortly.

Two quaternions are added component-wise. Put explicitly, two quaternions q and p add together to give

$$p + q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k}.$$

Similarly, following the defining multiplication laws above, p and q are multiplied to give

$$\begin{aligned} pq &= (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \\ &= p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 + (p_0q_1 + q_0p_1 + p2q3 - p3q2)\boldsymbol{i} + \cdots \\ &\quad \cdots + (p_3q_1 - p_1q_3 + p_0q_2 + q_0p_2)\boldsymbol{j} + (q_0p_3 + p_0q_3 + p_1q_2 - p_2q_1)\boldsymbol{k}. \end{aligned}$$

If we let $q = q_0 + \vec{q}$ and $p = p_0 + \vec{p}$, then we may express the product of quaternions in terms of the vector dot and cross products:

$$pq = p_0q_0 - \vec{p} \cdot \vec{q} + p_0\vec{q} + q_0\vec{p} + \vec{p} \times \vec{q}.$$

We define the *conjugate* of a quaternion q as

$$\bar{q} := q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$$

[3]This is later proven impossible by Hurwitz's Theorem.

and the *norm* $\|q\|$ by

$$\|q\|^2 := q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Famously, as one of the four normed division algebras, this norm operates as the usual Euclidean norm in $\mathbb{R}^4$, and so carries with it the usual properties such as $\|qp\| = \|q\|\,\|p\| = \|pq\|$ for all $p, q \in \mathbb{H}$. The quaternions of norm 1 are called the *unit* quaternions. When $q \neq 0$, we can show that

$$q^{-1} = \frac{\bar{q}}{\|q\|^2}.$$

We define the quaternionic exponential function by the standard Maclaurin series:

$$e^q = \sum_{m=0}^{\infty} \frac{q^m}{m!}.$$

This yields a sort of quaternionic Euler formula:

Proposition 1.1. *For each nonzero pure quaternion $\vec{q}$, the unit quaternion $e^{\vec{q}}$ is given by the formula*

$$e^{\vec{q}} = \cos\|\vec{q}\| + \mu \sin\|\vec{q}\|$$

where $\mu := \frac{\vec{q}}{\|\vec{q}\|}$ is the pure quaternion of unit length in the direction of $\vec{q}$.

Proof. Note that

$$\vec{q}^{\,2} = (q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) = -q_1^2 - q_2^2 - q_3^2 = -\|\vec{q}\|^2,$$

so that

$$\vec{q}^{\,2} = -\|\vec{q}\|^2, \quad \vec{q}^{\,3} = -\|\vec{q}\|^2\vec{q}, \quad \vec{q}^{\,4} = \|\vec{q}\|^4, \quad \vec{q}^{\,5} = \|\vec{q}\|^4\vec{q}, \quad \vec{q}^{\,6} = -\|\vec{q}\|^6, \quad \dots$$

Now, we can expand:

$$\begin{aligned}
e^{\vec{q}} &= \sum_{m=0}^{\infty} \frac{(\vec{q})^m}{m!} \\
&= 1 + \vec{q} - \frac{\|\vec{q}\|^2}{2!} - \frac{\|\vec{q}\|^2\vec{q}}{3!} + \frac{\|\vec{q}\|^4}{4!} + \frac{\|\vec{q}\|^4\vec{q}}{5!} - \frac{\|\vec{q}\|^6}{6!} + \cdots \\
&= \left(1 - \frac{\|\vec{q}\|^2}{2!} + \frac{\|\vec{q}\|^4}{4!} - \frac{\|\vec{q}\|^6}{6!} + \cdots\right) + \left(\vec{q} - \frac{\|\vec{q}\|^2\vec{q}}{3!} + \frac{\|\vec{q}\|^4\vec{q}}{5!} + \cdots\right) \\
&= \left(1 - \frac{\|\vec{q}\|^2}{2!} + \frac{\|\vec{q}\|^4}{4!} - \frac{\|\vec{q}\|^6}{6!} + \cdots\right) + \frac{\vec{q}}{\|\vec{q}\|}\left(\|\vec{q}\| - \frac{\|\vec{q}\|^3}{3!} + \frac{\|\vec{q}\|^5}{5!} + \cdots\right) \\
&= \cos(\|\vec{q}\|) + \frac{\vec{q}}{\|\vec{q}\|}\sin(\|\vec{q}\|).
\end{aligned}$$

■

Corollary 1.1.1. *For each nonzero quaternion $q = q_0 + \vec{q}$, we have the formula*

$$e^q = e^{q_0}\left(\cos\|\vec{q}\| + \mu \sin\|\vec{q}\|\right)$$

where $\mu := \frac{\vec{q}}{\|\vec{q}\|}$ is the pure quaternion of unit length in the direction of $\vec{q}$.

Proof. Note that common exponential property $e^q e^p = e^{q+p} = e^p e^q$ only holds provided that q and p commute. Since q_0 commutes with $\vec{q}$, we have that

$$e^q = e^{q_0} e^{\vec{q}} = e^{q_0}\left(\cos \|\vec{q}\| + \frac{\vec{q}}{\|\vec{q}\|} \sin \|\vec{q}\| \right) = e^{q_0}(\cos \|\vec{q}\| + \mu \sin \|\vec{q}\|).$$

∎

We will use the common notation $[g, h] = gh - hg$ to define the *commutator* of two elements. The commutator is equal to the additive identity if and only if g and h commute. Therefore, as noted in the previous proof $e^q e^p = e^{q+p} = e^p e^q$ only holds provided that $[q, p] = 0$.

Using Proposition 1.1, any quaternion $q = q_0 + \vec{q}$ with nonzero real part can then be written in an “extended polar form":

Proposition 1.2 (Quaternion polar form)**.** *If q is a quaternion with nonzero real and vector parts, then*

$$q = \|q\| \, e^{\mu\varphi} = \|q\| \, (\cos\varphi + \mu \sin\varphi)$$

where

$$\mu = \frac{\vec{q}}{\|\vec{q}\|} \qquad \text{and} \qquad \varphi = \tan^{-1} \frac{\|\vec{q}\|}{q_0}.$$

Proof. Let $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} = q_0 + \vec{q}$ be a quaternion with $q_0 \neq 0$ and $\vec{q} \neq \vec{0}$ as described and define the normalized “pure quaternion" part of q to be $\mu = \frac{\vec{q}}{\|\vec{q}\|}$. Then akin to Euler's formula in the complex case, we have that for any real number φ,

$$e^{\mu\varphi} = \cos\varphi + \mu \sin\varphi \tag{4}$$

using Proposition 1.1. Now, note that

$$q_0^2 + \|\vec{q}\|^2 = \|q\|^2 \iff \left(\frac{q_0}{\|q\|}\right)^2 + \left(\frac{\|\vec{q}\|}{\|q\|}\right)^2 = 1.$$

Therefore, by the Pythagorean theorem, for some $0 \leq \varphi \leq 2\pi$, we have both

$$\sin\varphi = \frac{q_0}{\|q\|} \qquad \text{and} \qquad \cos\varphi = \frac{\|\vec{q}\|}{\|q\|}.$$

In fact, since $\cos\varphi = \frac{\|\vec{q}\|}{\|q\|} > 0$, we know further that $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. This restricted domain satisfies

$$\varphi = \tan^{-1} \frac{\|\vec{q}\|}{q_0}.$$

So,

$$\begin{aligned} q &= q_0 + \vec{q} \\ &= \|q\| \left(\frac{q_0}{\|q\|} + \frac{\vec{q}}{\|\vec{q}\|} \cdot \frac{\|\vec{q}\|}{\|q\|} \right) \\ &= \|q\| \, (\cos\varphi + \mu \sin\varphi) \\ &= \|q\| \, e^{\mu\varphi}. \end{aligned}$$

∎

Quaternions are popularly used to represent rotations in $\mathbb{R}^3$ [8]. To see how this is possible, consider the pure quaternion p. Since the real part is zero, p can be considered a vector in $\mathbb{R}^3$; i.e., $p = \vec{p}$. Now, consider the operator

$$\Phi_q(p) = qpq^{-1}$$

where q is any non-zero quaternion. Replacing q with its polar representation, we have that

$$\Phi_q(p) = (\|q\| e^{\mu\varphi})p(\|q\|^{-1} e^{-\mu\varphi}) = e^{\mu\varphi} p e^{-\mu\varphi}$$

What may seem surprising is that $\Phi_q(p)$ represents a rotation of p (a vector) around the axis through μ of exactly 2φ radians. To gain some intuition, consider the simple direction $\mu = \boldsymbol{i}$. Then

$$\begin{aligned} e^{\boldsymbol{i}\varphi} p e^{-\boldsymbol{i}\varphi} &= e^{\boldsymbol{i}\varphi}(p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k})e^{-\boldsymbol{i}\varphi} \\ &= e^{\boldsymbol{i}\varphi}(p_1\boldsymbol{i} + (p_2 + p_3\boldsymbol{i})\boldsymbol{j})e^{-\boldsymbol{i}\varphi} \\ &= e^{\boldsymbol{i}\varphi}(p_1\boldsymbol{i})e^{-\boldsymbol{i}\varphi} + e^{\boldsymbol{i}\varphi}(p_2 + p_3\boldsymbol{i})\boldsymbol{j}e^{-\boldsymbol{i}\varphi} \\ &= e^{\boldsymbol{i}\varphi}(p_1\boldsymbol{i})e^{-\boldsymbol{i}\varphi} + e^{\boldsymbol{i}\varphi}(p_2 + p_3\boldsymbol{i})e^{\boldsymbol{i}\varphi}\boldsymbol{j} \\ &= p_1\boldsymbol{i} + e^{\boldsymbol{i}2\varphi}(p_2 + p_3\boldsymbol{i})\boldsymbol{j} \end{aligned}$$

Notice that the $\boldsymbol{i}$-component of p remains unchanged (since $\mu = \boldsymbol{i}$ is the axis of rotation), and the remaining terms are multiplied by a complex exponential with 2φ. Note also that if p did have a real part, then it would remain unchanged by Φ_q. Keeping these aspects in mind, it is easier to see that the mapping Φ defines a 2-to-1[4] surjective homomorphism between the Lie group $SU(2) = \{q \in \mathbb{H}; \|q\| = 1\}$ and the rotation group SO(3). This connection to rotations was found originally by Hamilton without modern language of linear algebra and morphisms [6]). Hence, if $p = p_0 + \vec{p}$ and $r = r_0 + \vec{r}$ are quaternions there there exists a q such that $\Phi_q(p) = r$ if and only if $\|\vec{p}\| = \|\vec{r}\|$ and $p_0 = r_0$.

1.3 Differential Operators

Consider the partial differential equation $f = 2f_{xy} - f_x$. Reordering, we see that might represent such an equation like $(1 - 2\partial_{xy} + \partial_x)f = 0$, so that the equation can be represented as $Q(f) = 0$ using the differential *operator* $Q = 1 - 2\partial_{xy} + \partial_x$. While you may think of the operator as shorthand notation for a differential equation, the structure of these operators are studied for their own sake.

Definition 1.1. [5] A *partial differential operator* (DO) is a polynomial in the symbols $\partial_{x_1}, \ldots, \partial_{x_n}$ over meromorphic functions of the variables $x_1, \ldots, x_n$:

$$P(\partial) = \sum_{\alpha_1=0}^{m_1} \cdots \sum_{\alpha_n=0}^{m_n} c_{\alpha_1 \cdots \alpha_n}(x_1, \cdots x_n)\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

[4] 2-to-1 because Φ_q and Φ_{-q} actually perform the same action since the negatives cancel.
[5] This is sometimes stated more succinctly using the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$: $\sum_{|\alpha| \le m} a_\alpha(x)\partial^\alpha$.

In the case of an *ordinary* differential operator Q, we will drop the subscripts so that

$$Q = \sum_{i=0}^{n} q_i(x)\partial^i.$$

Differential operators have a natural additive and multiplicative structure imposed upon them. Addition is performed by adding the coefficients of similar powers of ∂, while multiplication is defined in such a way that the resulting operator acts as a composition, i.e. so that for any two differential operators D and Q, we have that $D \circ Q(f) = D(Q(f))$. With this desire, we are forced to define pairwise multiplications as follows:

$$c_m(x)\partial^m \circ q_n(x)\partial^n = \sum_{i=0}^{m} \binom{m}{i} c_m(x) q_n^{(i)}(x)\partial^{m+n-i}.$$

This definition respects a sort of product rule so that $\partial \circ f(x) = f'(x) + f(x)\partial$. Unsurprisingly, the multiplication of differential operators are noncommutative. Therefore, just as for the quaternions, we will commonly refer to a commutator, which is 0 if and only if two differential operators D and Q happen to commute: $[D, Q] = D \circ Q - Q \circ D$.

To conclude, we wish to note some of the correspondences between differential operators and the "usual" polynomials as studied in seen in introductory calculus classes. For example, it is well known that if there exists some x_0 such that $p(x_0) = 0$, then we can factor p such that $p(x) = q(x)(x - x_0)$ for another polynomial q of lesser degree. Similarly, if a function f exists such that $D(f) = 0$, we can factor D such that $D = Q \circ \left(\partial - \frac{f'}{f}\right)$[22]. In fact, given any n-dimensional *space* of functions, we can construct a unique monic differential operator of degree n with that space as its kernel and any other operator that annihilates the same functions has this operator as a right factor [10]. This will be used in a proof to come.

2 Quaternion-Valued Solutions to ncKdV

2.1 Darboux Transformations for ncKdV

2.1.1 Isospectrality: From Matrices to Differential Operators

Consider the time-dependent matrix

$$M(t) = \begin{pmatrix} -t^2 + t + 2 & t^2 - t + 1 \\ t^2 - t & t^2 - t + 3 \end{pmatrix}.$$

Although the column vectors of M evolve in time independently and nonlinearly, the eigenvalues actually remain constant. This sort of property will be important in the Darboux transformation, so we will define it for matrices.

Definition 2.1. If the eigenvalues of a time-dependent matrix are constant, then we say that it is *isospectral.*

But how might you find $M(t)$ other than guessing randomly? Consider the following definition.

Definition 2.2. Two $n \times n$ matrices A and B are *similr* if there exists a matrix P such that $BP = PA$.

As you may have noticed, Definition (2.2) only differs slightly from the typical definition of *similar* as might be given in a linear algebra class[6]. It is well known that similar matrices share the same spectra. Interestingly, similr matrices nearly satisfy this property, even without the invertibility of P.

Proposition 2.1. *If A and B are similr, and λ is an eigenvalue for A with corresponding eigenvector $\vec{v}$, then $\vec{w} = P\vec{v}$ satisfies $B\vec{w} = \lambda\vec{w}$. λ is an eigenvalue for B only provided that $P\vec{v}$ is nonzero.*

Proof. Let λ be an eigenvalue for A with a corresponding eigenvector $\vec{v}$ so that $A\vec{v} = \lambda\vec{v}$. The applying matrix B to vector $P\vec{v}$ yields

$$B(P\vec{v}) = PA\vec{v} = P\lambda\vec{v} = \lambda(P\vec{v}).$$

However, λ is only an eigenvalue of B if the corresponding eigenvector $P\vec{v}$ is nonzero. ■

This property can be utilized to construct an isospectral matrix. Pick a matrix A with easily computed eigenvalues, say $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and some random time-dependent matrix $P(t)$, say $P(t) = \begin{pmatrix} 1 - t + t^2 & 1 \\ t^2 - t & 1 \end{pmatrix}$. Then according to Proposition 2.1, $B = P(t)AP(t)^{-1}$ has the same eigenvalues as A! In fact, $B = M(t)$, as defined above. So, although $M(t)$ looks complicated, it is really a constant matrix A that is “dressed" up by $P(t)$.

A seemingly random fact:

Lemma 2.2. *If $M(t)$ is isospectral, then $\dot{B} = [M, B] = MB - BM$ for some matrix M, where $\dot{B}$ is the derivative of B with respect to t.*

In particular, if $M(t) = P(t)AP(t)^{-1}$ then $M = \dot{P}P^{-1}$. So in some sense, differentiating isospectral matrices is like taking a commutator.

Like matrices, differential operators have *eigenfunctions* with corresponding eigenvalues. In other words, they can also be isospectral. In 1967, Gardner, Greene, Kruskal, and Miura discovered that if $u(x, t)$ was a soliton solution to the KdV, then the differential operator $L = \partial^2 + u(x, t)$ is isospectral, which they used to solve the KdV equation via the so-called *inverse scattering transform*[7] [18]. However, with more recent developments, we do not need to involve quantum theory. Recall taking the time derivative of a isospectral matrix is somehow analogous to a certain commutator. One year after [18], Peter Lax discovered that there exists a differential operator M such that L and M satisfies *Lax's equation* $\dot{L} = [M, L]$ if and only if $u(x, t)$ satisfies the KdV equation.

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[6] A and B are *similar* if there exists an invertible matrix P such that $B = PAP^{-1}$

[7] Although this approach is seen as outdated, it is a foundation for more modern methods. The solution u to a KdV equation is mapped to a potential in a Schrödinger equation, which is then solved by using initial scattering data.

Definition 2.3. Let P and L be time-dependent operators (or matrices) such that

$$\dot{L} = [P, L].$$

The P and L are called a *Lax pair*.

How might we find M? Let us first assume that M is an arbitrary differential operator of order 3, and let L remain as $L = \partial^2 + u(x,t)$. Then in order to satisfy a Lax equation, we need the coefficients of the differential operators on both sides to match up. Making minimal assumptions to satisfy this, we get that

$$\dot{L} = [M, L] \Longrightarrow u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}u_x u + c_1 u_x$$

for a constant c_1. Amazingly, it seems that if we take $c_1 = 0$, then $\dot{L} = [M, L]$ is precisely the KdV equation (15)! In particular, $M = \partial^3 + \frac{3}{2}u(x,t)\partial + \frac{3}{4}u_x(x,t)$.

What about if $u(x,t)$ was actually quaternion valued? Assuming that $M = \partial^3 + \frac{3}{2}u(x,t)\partial + \frac{3}{4}u_x(x,t)$, $\dot{L} = [M, L]$ becomes

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}u_x u + \frac{3}{4}u_x u, \tag{5}$$

what we will call the non-commutative KdV equation. How might we use this knowledge to find a previously unknown solution $u(x,t)$ to this equation?

Remark. *Notice if we had chosen a different M (i.e. not a third order differential operator), then we might have arrived a different equation at the end. This hierarchy of equations are referred to as the KdV hierarchy, of which KdV equation is only but one of the infinitely many other soliton equations. Similarly, the KdV hierarchy sits within another infinite family called the KP hierarchy, which come from expanding our class of operators to psuedo-differential operators. The KP hierarchy includes many other famous and arguably more important equations, all related by a natural geometry. To see how to derive the KdV equation from the perspective of the noncommutative KP hierarchy, see Subsection 7 in the Appendix.*

Recall our example with isospectral matrices. In that case, we had a simple matrix A, and we used a "dressing" trick to construct a complicated looking matrix with the same eigenvalues. Since L is isospectral, we may wonder if it is also a "dressed" version of some simpler constant differential operator.

Theorem 2.3. [a] *If $L = \partial^2 + u(x,t)$ for any KdV solution $u(x,t)$ then L is the unique solution to $L \circ K = K \circ \partial^2$ for some operator K.*

[a] K may be *psuedo*-differential operator, which I will avoid explaining.

Theorem 2.3 says that L is actually ∂^2, only dressed up. Essentially, the Darboux transformation exploits this isospectral behavior to generate new solutions via dressing, since if we can choose a K such that $L = \partial^2 + u(x,t)$, then $u(x,t)$ will automatically solve the KdV equation since L has the Lax pair shown above. However, in our isospectral matrices example, we had no restriction on what our resulting dressed "B" should look like, so we could pick any invertible $P(t)$. Now, we require that $L = \partial^2 + u(x,t)$, so we will have to proceed carefully in the next chapter. We will show that if K is chosen in a certain way, then L will always have the desired form. Then by choosing different $K's$, we will cycle through different ncKdV solutions.

2.1.2 The Darboux Transformation

Although the Darboux transformation used to generate solutions for ncKdV is close to the commutative case, some terms such as *span* must be suitably redefined to remain unambiguous.

Definition 2.4. Let $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ where $\phi_i : \mathbb{R}^b \to \mathbb{H}$ are quaternion-valued functions taking b real-valued inputs to a quaternion. The *span* of $\mathcal{F}$, denoted Span$\langle\mathcal{F}\rangle$, is the right $\mathbb{H}$-module generated by the elements of $\mathcal{F}$:

$$\text{Span}\langle\mathcal{F}\rangle = \left\{ \sum_{m=1}^{n} \phi_m \alpha_m \mid \alpha_m \in \mathbb{H} \right\}.$$

Definition 2.5. Let $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\}$. $\mathcal{F}$ is a *KdV-Darboux kernel* if each function $\phi \in \mathcal{F}$ satisfies the following properties:

1. $\phi_{xxx} = \phi_t$
2. $\phi_{xx} \in \text{Span}\langle\mathcal{F}\rangle$
3. The Wronski matrix $Wr(\mathcal{F})$ is invertible.[8]

Each KdV-Darboux kernel has a unique monic differential operator K whose kernel is exactly the span of $\mathcal{F}$. This formulation is extended to arbitrary endomorphisms acting on a unital associative algebra in [10].

Lemma 2.4. *If P is an ordinary differential operator in x whose coefficients depend on the additional parameter t and $f(x,t)$ is a quaternion-valued function, then*

$$\frac{\partial}{\partial t} P(f) = \dot{P}(f) + P(f_t).$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}\big(P(f)\big) &= \frac{\partial}{\partial t}\left(\sum_{m=1}^{n} c_m(x,t)\frac{\partial^m}{\partial x^m} f(x,t)\right) \\
&= \sum_{m=1}^{n} \frac{\partial}{\partial t}\left(c_m(x,t)\right)\frac{\partial^m}{\partial x^m} f(q,t) + \sum_{m=1}^{n} c_m(x,t)\frac{\partial}{\partial t}\frac{\partial^m}{\partial x^m} f(x,t) \\
&= \sum_{m=1}^{n} \frac{\partial}{\partial t}\left(c_m(x,t)\right)\frac{\partial^m}{\partial x^m} f(q,t) + \sum_{m=1}^{n} c_m(x,t)\frac{\partial^m}{\partial x^m}\frac{\partial}{\partial t} f(x,t) \\
&= \dot{P}(f) + P(f_t).
\end{aligned}$$

■

Theorem 2.5 provides a way to turn KdV-Darboux kernels into solutions of the KdV equation.

---

[8] Details of determining the invertibility of a quaternionic matrix is made precise in [17]

Theorem 2.5 (Quaternionic KdV-Darboux Construction)**.** *Suppose that $\mathcal{F}$ is a KdV-Darboux kernel and let K be the unique monic differential operator of order n with Span$\langle\mathcal{F}\rangle$ =* ker(K)*. Then there exists an operator L satisfying*

$$K \circ \partial^2 = L \circ K$$

such that $L = \partial^2 + u(x,t)$ where $u(x,t)$ is a solution to the non-commutative KdV equation.

Proof. Consider some arbitrary function $\phi \in \mathcal{F}$. Since $\mathcal{F}$ is a KdV-Darboux kernel, $\phi_{xx} \in$ Span$\langle\mathcal{F}\rangle$. Using this fact, we can show that ϕ is also in the kernel of $K \circ \partial^2$ since $K \circ \partial^2(\phi) = K(\phi_{xx}) = 0$ because $\phi_{xx} \in$ Span$\langle\mathcal{F}\rangle$ by definition of a KdV Darboux kernel (2.5), and thus is in the kernel of K by assumption. Then, by Theorem 5.1 in [10], $K \circ \partial^2 = L \circ K$ for some second order ordinary differential operator $L = \partial^2 + a(x,t)\partial + u(x,t)$. Writing

$$K = \partial^n + \sum_{m=1}^{n-1} \beta_m(x,t)\partial^m,$$

we can expand both sides of $K \circ \partial^2 = L \circ K$ to quickly reveal $a(x,t) = 0$ upon equating the highest order $k+1$ terms on both sides. Now that we know that $L = \partial^2 + u(x,t)$, we can substitute back into $K \circ \partial^2 = L \circ K$ to conclude that

$$K \circ \partial^2 = (\partial^2 + u(x,t)) \circ K. \tag{6}$$

Expanding the right hand side yields:

$$(\partial^2 + u(x,t)) \circ K = (\partial^2 + u(x,t)) \circ \left(\partial^n + \sum_{m=1}^{n-1} \beta_m(x,t)\partial^m\right)$$
$$= \partial^{n+2} + u(x,t)\partial^n + \sum_{m=1}^{n-1} \left[\big(\beta_m''(x,t) + u(x,t)\beta_m(x,t)\big)\partial^m + 2\beta_m'(x,t)\partial^{m+1} + \beta_m(x,t)\partial^{m+2}\right]$$
$$= K \circ \partial^2 + u(x,t)\partial^n + \sum_{m=1}^{n-1} \left[\big(\beta_m''(x,t) + u(x,t)\beta_m(x,t)\big)\partial^m + 2\beta_m'(x,t)\partial^{m+1}\right]$$

Subtracting $K \circ \partial^2$ from both sides of (6), we conclude that

$$u(x,t)\partial^n + \sum_{m=1}^{n-1} \left[\big(\beta_m''(x,t) + u(x,t)\beta_m(x,t)\big)\partial^m + 2\beta_m'(x,t)\partial^{m+1}\right] = 0.$$

By equating the n and $n-1$ order terms on both sides, we get

$$u(x,t) + 2\beta_{n-1}'(x,t) = 0 \tag{7}$$
$$\beta_{n-1}''(x,t) + u(x,t)\beta_{n-1}(x,t) + 2\beta_{n-2}'(x,t) = 0 \tag{8}$$

Let $\phi \in \mathcal{F}$. Now, differentiating $0 = K(\phi)$ with respect to t using Lemma 2.4,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} K(\phi) \\ &= \dot{K}(\phi) + K(\phi_t) \\ &= \dot{K}(\phi) + K(\phi_{xxx}) \\ &= \dot{K}(\phi) + K(\partial^3(\phi)) \\ &= \dot{K}(\phi) + K \circ \partial^3(\phi) \\ &= [\dot{K} + K \circ \partial^3](\phi). \end{aligned}$$

Since $[\dot{K} + K \circ \partial^3](\phi) = 0$, we can apply Theorem 5.1 from [10] again to conclude that $\dot{K} + K \circ \partial^3 = M \circ K$ for some third-order ordinary differential operator $M = \partial^3 + a(x,t)\partial^2 + b(x,t)\partial + c(x,t)$. As before, equating the highest order term $n+2$ from both sides quickly gives $a(x,t) = 0$. After algebra, we conclude that

$$\begin{aligned} \dot{K} = b(x,t)\partial^{n+1} + c(x,t)\partial^n + \sum_{m=1}^{n-1} \Big[&\big(\beta_m'''(x,t) + b(x,t)\beta_m'(x,t) + c(x,t)\beta_m(x,t)\big)\partial^m + \\ &+ \big(3\beta_m''(x,t) + b(x,t)\beta_m(x,t)\big)\partial^{m+1} + 3\beta_m'(x,t)\partial^{m+2}\Big]. \end{aligned}$$

Which, upon equating the n and $n+1$ terms yields

$$b(x,t) + 3\beta_{n-1}'(x,t) = 0 \tag{9}$$

$$c(x,t) + c(x,t)\beta_{n-1}(x,t) + 3\beta_{n-1}''(x,t) + 3\beta_{n-2}'(x,t) = 0 \tag{10}$$

Using (7), we can solve immediately for $b(x,t) = \frac{3}{2}u(x,t)$. Now, using this and the fact that

$$\beta_{n-2}'(x,t) = -\frac{1}{2}\left(\beta_{n-1}''(x,t) + u(x,t)\beta_{n-1}(x,t)\right)$$

from (8), we can substitute into (10):

$$\begin{aligned} c(x,t) &= -\frac{3}{2}u(x,t)\beta_{n-1}(x,t) - 3\beta_{n-1}''(x,t) - 3\left(-\frac{1}{2}\left(\beta_{n-1}''(x,t) + u(x,t)\beta_{n-1}(x,t)\right)\right) \\ &= -\frac{3}{2}\beta_{n-1}''(x,t) \\ &= -\frac{3}{2}\frac{\partial}{\partial x}\left(-\frac{1}{2}u(x,t)\right) \\ &= \frac{3}{4}u_x(x,t). \end{aligned}$$

Therefore $L = \partial^2 + u$ and $M = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_x$.

Using our previous value for $\dot{K}$, we can differentiate the left side of $K \circ \partial^2 = L \circ K$ with respect to t to give

$$\begin{aligned}\frac{\partial}{\partial t}\left(K \circ \partial^2\right) &= \dot{K} \circ \partial^2 \\ &= (M \circ K - K \circ \partial^3) \circ \partial^2 \\ &= M \circ K \circ \partial^2 - K \circ \partial^5 \\ &= M \circ L \circ K - L^2 \circ K \circ \partial\end{aligned}$$

while the right side becomes

$$\begin{aligned}\frac{\partial}{\partial t}(L \circ K) &= \dot{L} \circ K + L \circ \dot{K} \\ &= \dot{L} \circ K + L \circ (M \circ K - K \circ \partial^3) \\ &= \dot{L} \circ K + L \circ M \circ K - L^2 \circ K \circ \partial.\end{aligned}$$

We can subtract off $L^2 \circ K \circ \partial$ from both sides and perform right cancellation of K to conclude that

$$\dot{L} = [M, L].$$

So L satisfies a Lax equation. Using the fact that $\dot{L} = u_t$ and computing the commutator, we find that u satisfies

$$\begin{aligned}u_t &= [M, L] \\ &= M \circ L - L \circ M \\ &= \left(\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_x\right) \circ \left(\partial^2 + u\right) - \left(\partial^2 + u\right) \circ \left(\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u_x\right) \\ &= \frac{3}{4}uu_x + \frac{3}{4}u_x u + \frac{1}{4}u_{xxx}.\end{aligned}$$

■

This operation will be used throughout the rest of the paper, so we will provide notation and a definition.

Definition 2.6. Let Ω define the mapping from a KdV-Darboux kernel $\mathcal{F}$ to its corresponding KdV solution $u(x,t)$ obtained using Theorem 2.5. In other words, $\Omega(\mathcal{F}) = u(x,t)$. In the case that $\mathcal{F}$ is a singleton set containing only one function ϕ, let the notation $\Omega(\phi)$ be an abbreviation for $\Omega(\{\phi\})$.

Although this method connecting KdV-Darboux kernels to KdV solutions always works (as proven), our implementation within Mathematica remains computationally inefficient. Mysteriously, the following recursive quasi-determinants construction is much faster.

2.2 Quasideterminant Implementation of Darboux Transformations ★

The quasideterminant, introduced in 1991 by Gelfand and Retakh [20], is a replacement for determinants over noncommutative rings. If A is a square matrix, then the quasideterminant – or n^2 quasideterminants, since there is one defined for each entry in the matrix A – is a recursively constructed function that is defined if the submatrix A^{ij} is invertible over the ring.

The quasideterminant of a matrix A at position (i, j) is defined as

$$|A|_{ij} = a_{ij} - r_i(A)^{(j)}(A^{ij})^{-1}c_j(A)^{(i)}$$

where a_{ij} is the value of A and position (i, j), $r_i(A)^j$ is the i-th row of A exempting the j-th column, $c_j(A)^{(i)}$ is the j-th column of A exempting the i-th row, and A^{ij} is the submatrix of A formed by removing the i-th row and j-th column. Note that even in the case that the ring is commutative, the quasideterminant does not reduce to the traditional determinant, but to a ratio: $|A|_{ij} = (-1)^{i+j} \det A/ \det(A)^{ij}$. The next theorem comes from the results of [11].

Theorem 2.6. *Let u(x,t) be the ncKdV solution by the Darboux Transformation of the KdV-Darboux kernel* $\mathcal{F} = \{\phi_1, \ldots, \phi_n\}$ *and*

$$b_m = \left(\frac{\partial}{\partial x} W_m\right) W_m^{-1}, \quad W_m := |W(\phi_1, \ldots, \phi_m)|_{mm},$$

where $|X|_{ij}$ *denotes the quasideterminant of* X *at position* (i, j) *and* $W(\phi_1, \ldots, \phi_m)$ *denotes the Wronskian matrix generated from the enclosed functions* ϕ_i*. Then*

$$u(x, t) = 2\frac{\partial}{\partial x}\left(\sum_{m=1}^{n} b_m\right).$$

Proof. Let $K = \partial^n + v_1\partial^{n-1} + \ldots$ be the unique monic differential operator with $\mathcal{F}$ as its kernel. As shown in equation (7) of Theorem 2.5, we obtain $u = -2\partial v_1$ from the intertwining relationship $K \circ \partial^2 = (\partial^2 + u) \circ K$. By Theorem 1.1(ii) in [11] which decomposes $K = (\partial - b_n)(\partial - b_{n-1}) \cdots (\partial - b_1)$, we can write $v_1 = -\sum_m b_m$. Thus

$$u = -2\partial v_1 \qquad \Longrightarrow \qquad u(x, t) = 2\frac{\partial}{\partial x}\left(\sum_{m=1}^{n} b_m\right).$$

■

As mentioned previously, Theorem 2.6 provides a surprisingly quick computation of u given a KdV Darboux kernel. Figure 2.3 presents a possible Mathematica implementation of Theorem 2.6.

2.3 Relations between $\mathcal{F}$ and $u(x,t)$

Similar to the commutative case, multiplying each ϕ_i in the kernel on the right by some factor does not change which $u(x,t)$ the kernel $\mathcal{F}$ maps to since differential operators apply from left to right.

```
QKdVEGR[ker_] := Module[{Y, W, n, Q},
n  =  Length[ker];
Print["Step 1/6 (Constructing Y and W)"];
W = WronskianMatrix[ker];
Y =  W;
Y[[n, All]] = Map[D[#, x] &,  Y[[n, All]]];
Print["Step 2/6 (Simplifying Psuedo-Wronskian Y_NN)"];
Y = Simplify[qd[Y, {n, n}]];
Print["Step 3/6 (Simplifying Psuedo-Wronskian W_NN)"];
W = Simplify[qd[W, {n, n}]];
Print["Step 4/6 (Simplifying Inverse of W_NN)"];
W = Simplify[QInv[W]];
Print["Step 5/6 (Simplifying (Y_NN)*(W_NN)^(-1))"];
Q = Simplify[Qmult[Y, W]];
Print["Step 6/6 (Taking the Derivative)"];
2*D[Q, x]
]
qdinv[A_] := Table[QInv[qd[A, {j, i}]], {i, 1, Length[A]},
   {j, 1, Length[A]}]
qd[A_, {1, 1}] := A[[1, 1]] /;  Length[A] == 1
qd[A_, {i_, j_}] :=
 A[[i, j]] -
  (MatMult[MatMult[rij[i, j, A], qdinv[submatrix[i, j, A]]],
     cij[i, j, A]])[[1, 1]]
```

Figure 4: Mathematica implementation of Theorem 2.6. QKdVEGR calls upon the recursive quasideterminant function qd to form $\sum b_m$.

Proposition 2.7. *Let $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ be a KdV-Darboux kernel. Define $\widehat{\mathcal{F}} = \{\phi_1 q_1, \phi_2 q_2, \cdots, \phi_n q_n\}$ for any nonzero quaternions $q_1, q_2, \ldots, q_n \in \mathbb{H}$. Then $\Omega(\widehat{\mathcal{F}}) = \Omega(\mathcal{F})$.*

Proof. Let ϕ_m for $1 \leq m \leq n$ index the functions in the KdV-Darboux kernel $\mathcal{F}$. Similarly, let q_m denote the corresponding nonzero quaternion applied to ϕ_m in the construction of $\widehat{\mathcal{F}}$. Then there exists the unique monic differential operator K whose kernel is the span of $\mathcal{F}$. So $\phi_m \in \ker K$ by definition and $K(\phi_m) = 0$. But, each $\phi_m q_m \in \mathrm{Span}\langle \mathcal{F} \rangle = \ker K$. So $K(\phi_m q_m) = 0$ also. Thus the same unique K corresponds to both $\widehat{\mathcal{F}}$ and $\mathcal{F}$ (since their spans are the same). Then, since K corresponds to a unique solution $u(x,t)$ through the process in Theorem 2.5, $\Omega(\mathcal{F}) = \Omega(\widehat{\mathcal{F}})$. ■

Next, we conclude that multiplying each ϕ_i in $\mathcal{F}$ on the left by some nonzero quaternion merely rotates the associated solution $u(x,t)$.

Proposition 2.8. *Let $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ be a KdV-Darboux kernel and $\widehat{\mathcal{F}} = \{q\phi_1, q\phi_2, \cdots, q\phi_n\}$ where q is a nonzero quaternion. Then $q\Omega(\mathcal{F})q^{-1} = \Omega(\widehat{\mathcal{F}})$.*

Proof. Let K be the unique monic differential operator corresponding to $\mathcal{F}$, and $\widehat{K}$ be the unique monic differential operator corresponding to $\widehat{\mathcal{F}}$. First, I claim that $K = q^{-1}\widehat{K}q$. Letting ϕ_i be a general function in $\mathcal{F}$, we see that applying $q^{-1}\widehat{K}q$ to ϕ_i results in

$$q^{-1}\widehat{K}q(\phi_i) = q^{-1}\widehat{K}(q\phi_i) = q^{-1}(0) = 0$$

since $q\phi_i$ is in the kernel of $\widehat{K}$. Since $q^{-1}\widehat{K}q$ is a monic differential operator containing all of ϕ_i in its kernel, equality follows from uniqueness of K. Now, using the intertwining relationship,

$$\begin{aligned} K \circ \partial^2 = (\partial^2 + u) \circ K &\iff q^{-1}\widehat{K}q \circ \partial^2 = (\partial^2 + u) \circ q^{-1}\widehat{K}q \\ &\iff \widehat{K} \circ \partial^2 = q(\partial^2 + u) \circ q^{-1}\widehat{K} \\ &\iff \widehat{K} \circ \partial^2 = (\partial^2 + quq^{-1}) \circ \widehat{K}. \end{aligned}$$

Thus $\widehat{K}$ leads to the solution quq^{-1}, where u is the solution associated with K. Thus $q\Omega(\mathcal{F})q^{-1} = \Omega(\widehat{\mathcal{F}})$ as desired. ■

3 Three Basic Quaternion-Valued Solutions Types

In this section, we will pursue the characterization of three distinct solution types for the non-commutative KdV equation (5). While the solutions display similar properties to complexitons [12], study of quaternionic 1-solitons and their interactions with other solutions reveals a richer structure than anticipated in papers prior. Further, these quaternionic interactions inform a new understanding of the direct and inverse scattering problem in the real and complex case.

Definition 3.1. We say that the function $u(x,t)$ is a *soliton solution* of the ncKdV Equation (5) if all of the elements in the KdV-Darboux kernel are of the form $\alpha e^{qx+q^3t} + \beta e^{-qx-q^3t}$ for some $\alpha, \beta \in \mathbb{H}$ and for $q \in \mathbb{C}$ with nonzero real part.

Note that this definition of a soliton is uncharacteristically broad – we are simply including all localized solutions, even if it is singular or does not maintain a consistent shape. In particular, we want to include the so-called *breather* solitons, which are localized waves that do not necessarily maintain the exact same profile during translation.

3.1 Trivial Solutions

Consider the case where each of the functions $\phi_i \in \mathcal{F}$ is of the form αe^{qx+q^3t} for $\alpha, q \in \mathbb{H}$. While $\mathcal{F}$ is a KdV-Darboux kernel according to Definition 2.5, its associated solution is $u(x,t) = 0$ the trivial solution. Traditionally, this precludes further investigation in the real and complex cases since their inclusion in a KdV-Darboux kernel does not lead to a higher ordered soliton interaction. However, as we will see, adding functions such as these to a kernel can change the solution significantly for reasons that help explain n-soliton interactions.

Since constant multiples of e^{qx+q^3t} just construct the trivial solution, this is often referred to as the trivial (“vacuum") eigenfunction. This function can be used to construct other more interesting functions that satisfy the properties for functions in a KdV-Darboux kernel (2.5), as will be seen in sections to come.

3.2 Rational Solutions

Much of this discussion is expounded upon in more detail in Albert Serna’s master thesis. Since partial differentiation commutes (by Clairaut’s theorem), one way we can create more balanced functions as defined in Definition (2.5) is as follows:

Definition 3.2. For $m \in \mathbb{N}$ define Δ_m to be the polynomial

$$\Delta_m = \frac{\partial^m}{\partial q^m} e^{qx+q^3t}\Big|_{q=0}.$$

Δ_m is balanced for $m \in \mathbb{N}$. Additionally, since property (1) in the KdV-Darboux kernel definition is linear, we can also take linear combinations of these functions and satisfy the same property.

Definition 3.3. Let

$$\mathcal{B} = \left\{ \sum_{i=0}^{m} \Delta_i \alpha_i : \alpha_m \in \mathbb{H} \right\}$$

be the set of polynomials over the functions Δ_i as defined in Definition (3.2).

Any polynomial in $\mathcal{B}$ is also balanced. However, in order satisfy the closure property and construct a KdV-Darboux kernel, we must include all of its even ordered nonzero derivatives.

Definition 3.4. For any $\Phi \in \mathcal{B}$ then let $\mathcal{F}(\Phi)$ be defined as

$$\mathcal{F}(\Phi) = \{\Phi, \Phi_{xx}, \Phi_{xxxx}, \ldots, \Phi_M\}$$

where

$$M = \begin{cases} M = m - 1, & m \text{ is odd} \\ M = m, & m \text{ is even} \end{cases}.$$

In order to prove that this is indeed a KdV-Darboux kernel, the following properties are useful.

Lemma 3.1. *Let Δ_m be as described in Definition (3.2). Then*

$$\frac{\partial^p}{\partial x^p} \Delta_m = m(m-1)\cdots(m-p+1)\Delta_{m-p} = \frac{m!}{(m-p)!}\Delta_{m-p}.$$

Proof. By commuting partial derivatives,

$$\frac{\partial^p}{\partial x^p} \Delta_m = \frac{\partial^m}{\partial q^m} q^p e^{qx+q^3 t}\Big|_{q=0}.$$

Conceiving of the differential operator $\partial^m = \frac{\partial^m}{\partial q^m}$, we can think of the above computation as

$$\left(\partial^m \circ q^p e^{qx+q^3 t}\right)\Big|_{q=0} = \left((\partial^m \circ q^p) \circ e^{qx+q^3 t}\right)\Big|_{q=0}.$$

However, computing $\partial^m \circ q^p$ gives us

$$\partial^m \circ q^p = \sum_{i=0}^{m} \binom{m}{l} \frac{\partial^l}{\partial q^l}(q^p)\, \partial^{m-l}.$$

The only nonzero term not containing a q, and thus the only term that remains after evaluating at 0, occurs at $i = p$ when $\frac{\partial^p}{\partial q^p} q^p = p!$. Therefore

$$\begin{aligned} \left((\partial^m \circ q^p) \circ e^{qx+q^3 t}\right)\Big|_{q=0} &= \left(\binom{m}{p} p! \partial^{m-p} \circ e^{kx+k^3 t}\right)\Big|_{q=0} \\ &= \left(\frac{m!}{(m-p)!} \frac{\partial^{m-p}}{\partial q^{m-p}} e^{kx+k^3 t}\right)\Big|_{q=0} \\ &= \frac{m!}{(m-p)!}\Delta_{m-p} \end{aligned}$$

■

Corollary 3.1.1. *Let Δ_m be as described in Definition (3.2). Then*

$$\frac{\partial^p}{\partial t^p}\Delta_m = \frac{m!}{(m-3p)!}\Delta_{m-3p}.$$

Proof. This follows from the previous lemma since Δ_m satisfies property (1) of Definition (2.5). ■

Remark. *Putting the previous two results together, we note that for $p \in \mathbb{N}$,*

$$\frac{\partial^p}{\partial t^p}\Delta_m = \frac{\partial^{3p}}{\partial t^{3p}}\Delta_m.$$

Proposition 3.2. *$\mathcal{F}(\Phi)$ is a KdV-Darboux kernel.*

Proof. Clearly, $\mathcal{F}(\Phi)$ is closed under second derivatives by construction. Now we must show that any $\phi \in F(\Phi)$ satisfies $\phi_{xxx} = \phi_t$. An arbitrary $\phi \in F(\Phi)$ has the form

$$\phi = \frac{\partial^{2n}}{\partial x^{2n}} \sum_{i=0}^{m} \Delta_i a_i$$

for some $M < n \in \mathbb{N}$. Taking the derivative with respect to t,

$$\phi_t = \frac{\partial^{2n}}{\partial x^{2n}} \sum_{i=0}^{m} \left(\frac{\partial}{\partial t}\Delta_i\right) a_i = \frac{\partial^{2n}}{\partial x^{2n}} \sum_{i=0}^{m} \left(\frac{\partial^3}{\partial x^3}\Delta_i\right) a_i = \phi_{xxx}.$$

■

Using results that will be presented later in the paper, this form will be improved upon in Section 4.3 as an application of the direct scattering problem solution. In particular, it is shown that one only needs to consider polynomials in $\mathcal{B}$ of odd order.

3.3 1-Solitons

Here we will consider the solitons constructed from singleton KdV-Darboux kernels $\mathcal{F} = \{\phi\}$ and their associated properties. Using Theorem 2.6, the general solution of such a construction is

$$u(x,t) = \Omega(\phi) = 2(\phi_x\phi^{-1})_x = 2(\phi_{xx}\phi^{-1} - (\phi_x\phi^{-1})^2). \tag{11}$$

3.3.1 General Form of ϕ

Consider quaternionic functions of the form $\phi = \sum_{i=1}^{n} \alpha_i e^{q_i x + q_i^3 t}$ ($\alpha_i, q_i \in \mathbb{H}$). As noted previously, ϕ is balanced in the sense of property (1) in Definition (2.5) due to its linearity. However, in order for its second derivative to be in the span of the singleton kernel $\mathcal{F}$, we need the additional restriction that $\phi_{xx} = \phi q$ for some $q \in \mathbb{H}$. Therefore given a choice of q, our general ϕ for the generation of a 1-soliton is of the form

$$\phi(q,\alpha,\beta) = \alpha e^{qx+q^3t} + \beta e^{-qx-q^3t}. \tag{12}$$

Further, it is sufficient to only consider $q \in \mathbb{C} \subset \mathbb{H}$, as demonstrated in the proposition below.

Proposition 3.3. *Given a ϕ as in equation (12) such that $\Omega(\phi) = u(x,t)$, another function $\widehat{\phi}$ of the same form can be found such that $\widehat{q} \in \mathbb{C}$ and $\Omega(\widehat{\phi}) = u(x,t)$.*

Proof. Define some $\phi = \alpha e^{qx+q^3t} + \beta e^{-qx-q^3t}$ with $\alpha, \beta, q \in \mathbb{H}$. As introduced in section 1.2, the operator $\Phi_\gamma(q) = \gamma q \gamma^{-1}$ effects the rotation of the vector part $\vec{q}$ of q about $\gamma \in \mathbb{H}$. Choose $\gamma \in \mathbb{H}$ such that $\Phi_\gamma(q)$ rotates $\vec{q}$ onto the $\boldsymbol{i}$-axis; that is, such that $\Phi_\gamma(q) = q_0 + \|\vec{q}\|\,\boldsymbol{i} = \widehat{q}$. Then the inverse rotation $(\Phi_\gamma(q))^{-1} = \Phi_{\gamma^{-1}}(\widehat{q}) = \gamma^{-1}\widehat{q}\gamma$ rotates the complex number $\widehat{q}$ onto q. So,

$$\begin{aligned}
\phi &= \alpha e^{(\gamma^{-1}\widehat{q}\gamma)x+(\gamma^{-1}\widehat{q}\gamma)^3t} + \beta e^{-(\gamma^{-1}\widehat{q}\gamma)x-(\gamma^{-1}\widehat{q}\gamma)^3t} \\
&= \alpha e^{\gamma^{-1}(\widehat{q}x+\widehat{q}^3t)\gamma} + \beta e^{\gamma^{-1}(-\widehat{q}x-\widehat{q}^3t)\gamma} \\
&= \alpha\gamma^{-1}e^{\widehat{q}x+\widehat{q}^3t}\gamma + \beta\gamma^{-1}e^{\widehat{q}x+\widehat{q}^3t}\gamma.
\end{aligned}$$

Using Proposition 2.7, we conclude that $\widehat{\phi} = \phi\gamma^{-1} \mapsto u(x,t)$. ■

From here on, we will only consider $q = q_0 + q_1\boldsymbol{i}$ such that $q_0 \neq 0$ since otherwise, e^q would decompose into purely trigonometric functions and ϕ would not construct a localized solution. Without loss of generality, assume that $q_0 > 0$.

3.3.2 Properties of 1-Solitons

This section will characterize the properties of 1-Solitons and link them back to the parameters of ϕ. We will start by showing when ϕ leads to the trivial solution $u(x,t) = 0$.

Proposition 3.4. *Given a ϕ of the form in equation (12) with $q_0 > 0$, $\Omega(\phi) = 0$ if either α or β are zero.*

Proof. If $\alpha\beta = 0$, then either α or β is zero. Let $\alpha = 0$ first. Then $\phi = \beta e^{-qx-q^3t}$. According to equation (11) and using $\phi_x = -\beta q e^{-qx-q^3t} = -q\phi$, we have that

$$\Omega(\phi) = u(x,t) = 2(\phi_x\phi^{-1})_x = 2(-q\phi\phi^{-1})_x = 2(-q)_x = 0.$$

The same argument holds for $\beta = 0$. ■

Traditionally, soliton velocity might be found by determining a form for $u(x,t)$ like $u(x+ct)$, where c represents how many units the soliton moves per unit time. However, a quaternionic 1-soliton can be thought to have two associated velocities: one for the *envelope*, and one for the encased periodic curves. Note that by equation (11), this question can be investigated by studying $\phi(x,t)$ alone.[9] Both of the speeds can be determined by the more general decomposition of ϕ outlined over the next few results.

Lemma 3.5. *Let $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} = q_0 + \vec{q}$ be an arbitrary quaternion. Then for any imaginary unit $\gamma \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, we have*

$$e^q = \frac{\|\vec{q}\| - \vec{q}\gamma}{2\|\vec{q}\|}e^{q_0+\gamma\|\vec{q}\|} + \frac{\|\vec{q}\| + \vec{q}\gamma}{2\|\vec{q}\|}e^{q_0-\gamma\|\vec{q}\|}.$$

[9]Since equation (11) shows that if $\phi(x+ct)$, then $u(x+ct)$.

Proof. As discussed previously in Section 1.2, we know that

$$e^{\vec{q}} = \cos(\|\vec{q}\|) + \frac{\vec{q}}{\|\vec{q}\|}\sin(\|\vec{q}\|).$$

Let γ be either $\boldsymbol{i}, \boldsymbol{j}$, or $\boldsymbol{k}$. Then since $\gamma^2 = -1$, Euler's formula $e^{\gamma\theta} = \cos(\theta) + \gamma\sin(\theta)$ holds for each imaginary unit individually. Now, note that

$$\begin{aligned} -\frac{\gamma}{2}(e^{\gamma\|\vec{q}\|} - e^{-\gamma\|\vec{q}\|}) &= -\frac{\gamma}{2}\left[\cos(\|\vec{q}\|) + \gamma\sin(\|\vec{q}\|) - (\cos(\|\vec{q}\|) - \gamma\sin(\|\vec{q}\|))\right] \\ &= -\gamma^2\sin(\|\vec{q}\|) \\ &= \sin(\|\vec{q}\|) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}(e^{\gamma\|\vec{q}\|} + e^{-\gamma\|\vec{q}\|}) &= \frac{1}{2}\left[\cos(\|\vec{q}\|) + \gamma\sin(\|\vec{q}\|) + \cos(\|\vec{q}\|) - \gamma\sin(\|\vec{q}\|)\right] \\ &= \cos(\|\vec{q}\|). \end{aligned}$$

We can substitute these in to get:

$$\begin{aligned} e^{q} &= e^{q_0+\vec{q}} \\ &= e^{q_0}e^{\vec{q}} \\ &= e^{q_0}\left[\cos(\|\vec{q}\|) + \frac{\vec{q}}{\|\vec{q}\|}\sin(\|\vec{q}\|)\right] \\ &= e^{q_0}\left[\frac{1}{2}\left(e^{\gamma\|\vec{q}\|} + e^{-\gamma\|\vec{q}\|}\right)\right) + \frac{\vec{q}}{\|\vec{q}\|}\left(-\frac{\boldsymbol{i}}{2}\left(e^{\gamma\|\vec{q}\|} - e^{-\gamma\|\vec{q}\|}\right)\right] \\ &= e^{q_0}\left[\frac{\|\vec{q}\| - \vec{q}\,i}{2\,\|\vec{q}\|}e^{\gamma\|\vec{q}\|} + \frac{\|\vec{q}\| + \vec{q}\,i}{2\,\|\vec{q}\|}e^{-\gamma\|\vec{q}\|}\right] \\ &= \frac{\|\vec{q}\| - \vec{q}\gamma}{2\,\|\vec{q}\|}e^{q_0+\gamma\|\vec{q}\|} + \frac{\|\vec{q}\| + \vec{q}\gamma}{2\,\|\vec{q}\|}e^{q_0-\gamma\|\vec{q}\|} \end{aligned}$$

∎

Proposition 3.6. *Let $q = q_0 + \vec{q}$ be an arbitrary quaternion, where $\vec{q} = q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$. Also, let $\alpha = q_0(x + (q_0^2 - 3\,\|\vec{q}\|^2)t)$, and $\beta = x + (3q_0^2 - \|\vec{q}\|^2)t$. Then*

$$e^{qx+q^3t} = \frac{\|\vec{q}\| - \vec{q}\,\boldsymbol{i}}{2\,\|\vec{q}\|}e^{\alpha+\boldsymbol{i}\|\vec{q}\|\beta} + \frac{\|\vec{q}\| + \vec{q}\,i}{2\,\|\vec{q}\|}e^{\alpha-\boldsymbol{i}\|\vec{q}\|\beta}$$

for all $x, t \in \mathbb{R}$.

Proof. Note that since $\vec{q}^{\,2} = -\|\vec{q}\|^2$, we have that

$$(q_0 + \vec{q})^3 = q_0^3 + 3q_0^2\vec{q} - 3q_0\,\|\vec{q}\|^2 - \|\vec{q}\|^2\,\vec{q}.$$

So

$$\begin{aligned} e^{qx+q^3t} &= e^{(q_0+\vec{q})x+(q_0+\vec{q})^3t} \\ &= e^{(q_0+\vec{q})x+q_0^3t+3q_0^2\vec{q}t-3q_0\|\vec{q}\|^2t-\|\vec{q}\|^2\vec{q}t} \\ &= e^{(q_0x+q_0^3t-3q_0\|\vec{q}\|^2t)+\vec{q}(x+3q_0^2t-\|\vec{q}\|^2t)} \\ &= e^{\alpha+\vec{q}\beta}. \end{aligned}$$

Now, choosing γ to be the imaginary unit $\boldsymbol{i}$, we can use the previous theorem to conclude that

$$e^{\alpha+\vec{q}\beta} = \frac{\|\vec{q}\| - \vec{q}\boldsymbol{i}}{2\|\vec{q}\|}e^{\alpha+\boldsymbol{i}\|\vec{q}\|\beta} + \frac{\|\vec{q}\| + \vec{q}\boldsymbol{i}}{2\|\vec{q}\|}e^{\alpha-\boldsymbol{i}\|\vec{q}\|\beta}.$$

∎

By letting $q_2 = q_3 = 0$, we see that the "envelope part" $e^a = e^{q_0(x+(q_0^2-3q_1^2)t)}$ can be written as a function of $x+(q_0^2-3q_1^2)t$, and the "periodic part" $e^{\beta\boldsymbol{i}} = e^{\boldsymbol{i}(x+(3q_0^2-\|\vec{q}\|^2)t)}$ can be written as a function of $x+(3q_0^2-q_1^2)t$. This shows that the envelope part of $u(x,t)$ has a velocity of $(q_0^2-3q_1^2)$ while the periodic part has a velocity of $(3q_0^2-q_1^2)$.

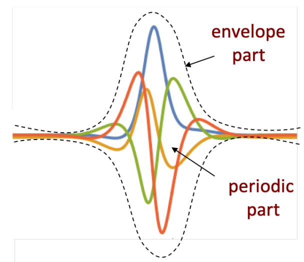

Figure 5: An example quaternionic 1-soliton solution to the ncKdV equation with its envelope and periodic part labeled.

Definition 3.5. For a given $\phi = \alpha e^{qx+q^3t} + \beta e^{-qx-q^3t}$ with $q = a + b\boldsymbol{i}$ mapping to a soliton solution of the ncKdV equation $\Omega(\phi) = u(x,t)$, we will say the the *envelope velocity* of $u(x,t)$ is $v = a^2-3b^2$, while the *periodic velocity* of $u(x,t)$ is given by $3a^2-b^2$.

If the velocity is negative, then it will move in the positive direction at a speed proportional to its magnitude. Inversely, if the velocity is positive, then it will be moving in the negative direction at a speed proportional to its magnitude.

Like in the real case, we have that $\lim_{x\to\pm\infty} u = 0$. Unlike the real case however, there can be a difference at the left and the right. More specifically, if one normalizes a general 1-soliton $u(x,t)$, then the periodic parts at the left and right of the localized hump have different behaviors. An explanation of this comes from considering $\phi = \alpha e^{qx+q^3t} + \beta e^{-qx-q^3t}$ as $x, t \to \pm\infty$. A technique such as this will be used to develop a more general theory that allows the decomposition of $u(x,t)$ into asymptotic left and right parts in Section 4.

Another important characteristic of ϕ is its associated *center*. This corresponds to the "middle" of the envelope part of the soliton. Using that fact that $\Omega(\phi) = 2(\phi_x\phi^{-1})_x$, we can see that the maximum amplitude of the soliton will occur when $|\phi|$ is minimized so that $|\Omega(\phi)|$ is maximized. By minimizing ϕ, and ignoring small trigonometric terms, we find that the center is at

$$c_\phi(t) = \frac{\ln|\alpha^{-1}\beta|}{2a} - vt, \tag{13}$$

where $v = a^2-3b^2$, the velocity of the envelope. This matches the form for the center in the real case shown in [3] since if we were to make q and β real and $\alpha = 1$ such as in the paper, then (13) reduces to

$$\frac{\ln|\beta|/2}{a} - a^2t$$

4 The Direct Scattering Problem

The direct and indirect "scattering problem" is terminology traditionally associated with the renowned first method to solve the KdV equation published by Gardner, Greene, Kruskal, and Miura in 1967 [18]. In this paper, the direct scattering problem refers to how the properties of individual solitons within a multisoliton solutions affect their interaction.

In the real case, two solitons as collide in a particle-like fashion where they seem to exchange amplitudes and velocities. This interaction is described by the famous *phase shift*, where one soliton is shifted forward and the other shifted back as a result of interaction [19]. This shift in the real case is found the be wholly contingent on the velocities involved. As we will see, in the quaternion case, this phase shift is also dependent on the quaternionic coefficients, which can be adjusted to allow for forward shifts, backward shifts, and solutions that do not interact at all. Surprisingly, the following theorem about how the trivial solutions affect the KdV-Darboux transformation is of use.

4.1 The Trivial Solution Shift

Lemma 4.1. *Suppose $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ is a KdV-Darboux kernel and that Q is a constant coefficient differential operator. Then as long as $Q(\mathcal{F}$ has an invertible Wronskian, $Q(\mathcal{F}) = \{Q(\phi_1), \ldots, Q(\phi_n)\}$ is also a KdV-Darboux kernel.*

Proof. Since Q is a constant coefficient differential operator of some order n, it has the form $Q = \sum_{i=0}^{n} a_i \partial^i$ where $a_i \in \mathbb{H}$. Therefore applying Q to some $\phi \in \mathcal{F}$ is equivalent to taking some linear combination of the derivatives of ϕ:

$$Q(\phi) = a_1\phi + a_2\phi_x + \cdots + a_n\phi_{x^n}.$$

Since partial derivatives commute, property 1 for a KdV-Darboux kernel (2.5) is still satisfied. All that remains is showing that property 2 holds. Since derivatives distribute, we see that $Q(\phi)_{xx} = Q(\phi_{xx})$. Now, since $\phi_{xx} \in \mathcal{F}$, there exists some linear combination of other functions $f_i \in \mathcal{F}$ such that $\phi_{xx} = \sum_{i=0}^{m} f_i a_i$. Noting that for any functions g and h, $Q(g)+Q(h) = Q(g+h)$, we see that

$$Q(\phi_{xx}) = Q\left(\sum_{i=0}^{m} f_i a_i\right) = \sum_{i=0}^{m} Q(f_i) a_i.$$

∎

Lemma 4.2. *Suppose $\mathcal{F} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ is a KdV-Darboux kernel. Define*

$$G = \{g_i \mid g_i = \alpha_i e^{q_i x + q_i^3 t} \text{ for } 0 < i \leq N\}.$$

Let Q be the unique monic constant coefficient differential operator with Span$\langle G\rangle$ as its kernel. Then as long as the Wronskian of the bigger set $\mathcal{F} \cup G$ is invertible, it is also a KdV-Darboux kernel and $\Omega(\mathcal{F} \cup G) = \Omega(Q(\mathcal{F}))$.

Proof. Let K be the unique monic differential operator corresponding to $\mathcal{F} \cup G$, and $\widehat{K}$ be the unique monic differential operator corresponding to $Q(\mathcal{F})$. First, I claim that $K = \widehat{K} \circ Q$. Letting ϕ be a general function in $\mathcal{F} \cup G$. Then either $\phi \in \mathcal{F}$ or $\phi \in G$. If $\phi \in G$, then applying $\widehat{K} \circ Q(\phi) = \widehat{K} \circ 0 = 0$ since Q is the KdV-Darboux kernel of G. Otherwise, if $\phi \in \mathcal{F}$, then

$\widehat{K} \circ Q(\phi) = \widehat{K}(Q(\phi)) = 0$ since $Q(\phi) \in Q(\mathcal{F})$, and $\widehat{K}$ is the KdV-Darboux kernel of $Q(\mathcal{F})$. Since $\widehat{K} \circ Q$ is a monic differential operator containing all of functions of $\mathcal{F} \cup G$ in its kernel, equality follows from uniqueness of K. Now, using the intertwining relationship,

$$\begin{aligned} K \circ \partial^2 = (\partial^2 + u) \circ K &\iff \widehat{K} \circ Q \circ \partial^2 = (\partial^2 + u) \circ \widehat{K} \circ Q \\ &\iff (\widehat{K} \circ \partial^2) \circ Q = ((\partial^2 + u) \circ \widehat{K}) \circ Q \\ &\iff \widehat{K} \circ \partial^2 = (\partial^2 + u) \circ \widehat{K}. \end{aligned}$$

Thus $\widehat{K}$ gives the same solution u associated with K. Thus $\Omega(Q(\mathcal{F})) = \Omega(\widehat{\mathcal{F}} \cup G)$ as desired. ■

So if there are trivial eigenfunctions in a KdV-Darboux kernel $\mathcal{F}$, the associated solution $u(x, t)$ is the same one associated with the smaller kernel created by applying Q to each non-trivial function. This is important because asymptotically, the form of our general 1-soliton behaves like a trivial eigenfunction within $\mathcal{F}$ since

$$\alpha e^{qx+q^3t} + \beta e^{-qx-q^3t} \approx \alpha e^{qx+q^3t} \text{ as } x \text{ and } t \text{ get large,}$$

and

$$\alpha e^{qx+q^3t} + \beta e^{-qx-q^3t} \approx \beta e^{-qx-q^3t} \text{ as } x \text{ and } t \text{ get small.}$$

This gives rise to the following theorem.

Theorem 4.3. *Let $\mathcal{F} = \{\phi_1, \ldots, \phi_n\}$ and $\widehat{\phi} \notin$ Span $\langle \mathcal{F} \rangle$ where*

$$\widehat{\phi} = \alpha e^{qx+q^3t} + \beta e^{-qx-q^3t}$$

and $q = a + b\boldsymbol{i}$ with $a > 0$. If $u(x, t) = \Omega(\mathcal{F} \cup \{\widehat{\phi}\})$, then u converges uniformly in time to $u_+(x, t) = \Omega(\mathcal{F}_+)$ as $x \to \infty$ and uniformly in time to $u_-(x, t) = \Omega(\mathcal{F}_-)$ as $x \to -\infty$, where

$$\mathcal{F}_+ = \{Q_+(\phi_1), \cdots, Q_+(\phi_n)\} \qquad Q_+ = \partial - \alpha q \alpha^{-1}$$

and

$$\mathcal{F}_- = \{Q_-(\phi_1), \cdots, Q_-(\phi_n)\} \qquad Q_- = \partial + \beta q \beta^{-1}.$$

In other words, given any $\epsilon > 0$ we may choose a distance δ relative to the moving center $c_{\widehat{\phi}}(t)$ so that for all t, if x satisfies $|c_{\widehat{\phi}}(t) - x| > \delta$, then either $|u(x, t) - u_+(x, t)| < \epsilon$ when $x > c_{\widehat{\phi}}(t)$ or $|u(x, t) - u_-(x, t)| < \epsilon$ when $x < c_{\widehat{\phi}}(t)$.

Proof. We first show that for any particular time $t_0 \in \mathbb{R}$, $u(x, t_0)$ approaches $u_+(x, t_0)$ and $u_-(x, t_0)$ as $x >> c_{\widehat{\phi}}(t_0)$ and $x << c_{\widehat{\phi}}(t_0)$, respectively. Note that the center equation $c_{\widehat{\phi}}(t_0)$ corresponds exactly to the point at which the two exponential terms of $u(x, t_0)$ are in equilibrium in the sense that

$$\left|\alpha e^{qc_{\widehat{\phi}}(t_0)+q^3t_0}\right| = \left|\beta e^{-qc_{\widehat{\phi}}(t_0)-q^3t_0}\right|.$$

However, as $x >> c_{\widehat{\phi}}(t_0)$, the term $\alpha e^{qx+q^3t_0}$ dominates, so that $\left|\alpha e^{qx+q^3t_0}\right| >> \left|\beta e^{-qx-q^3t_0}\right|$. So the approximation $\widehat{\phi} \approx \widehat{\phi}_+ = \alpha e^{qx+q^3t_0}$ becomes increasingly sharp, as does

$$u(x, t_0) = \Omega(\mathcal{F} \cup \{\widehat{\phi}\}) \approx \Omega(\mathcal{F} \cup \{\widehat{\phi}_+\}).$$

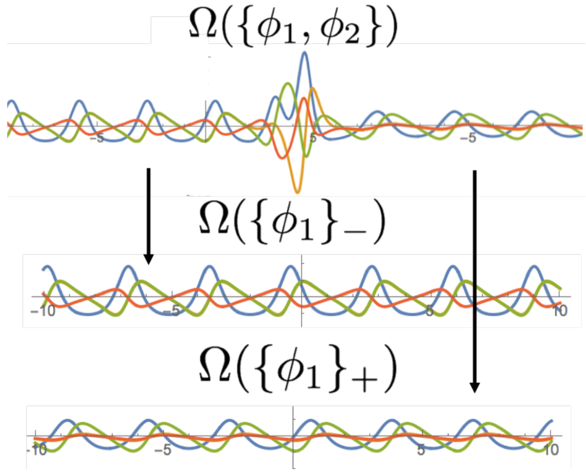

Figure 6: A example quaternionic 2-soliton solution to the ncKdV equation depicted before collision and after collision with centers labeled according to Definition (4.1).

Now, by to the proceeding Lemma 4.2,

$$\Omega(\mathcal{F} \cup \{\widehat{\phi}_+\}) = \Omega(Q_+(\mathcal{F})) = \Omega(\mathcal{F}_+) = u_+(x,t),$$

where $Q_+ = \partial - \alpha q \alpha^{-1}$ is the computed unique monic constant coefficient differential operator with Span$\langle G \rangle$. Thus, as long as x is chosen to be adequately larger than $c_{\widehat{\phi}}(t_0)$, $u(x,t_0)$ can be made arbitrarily close to $u_+(x,t_0)$. This means that for every $\epsilon > 0$, there exists a $\delta_+ > 0$ so that $c_{\widehat{\phi}}(t) - x > \delta_+$ implies $|u(x,t_0) - u_+(x,t_0)| < \epsilon$. By a similar argument, for every $\epsilon > 0$, there exists a $\delta_- > 0$ so that $x - c_{\widehat{\phi}}(t) > \delta_-$ implies that $|u(x,t_0) - u_-(x,t_0)| < \epsilon$. Now, by choosing $\delta = \min\{\delta_-, \delta_+\}$, the claim is true for fixed time t_0.
Now, all that is left to show is that the choice of δ is independent of t. It suffices to show that if t varies by some amount, then x can be shifted by this same amount so that the magnitude of the approximate solutions $\left|\alpha e^{qx+q^3 t}\right|$ and $\left|\beta e^{-qx-q^3 t}\right|$ remain the same. This guarantees that as the solution $u(x,t)$ moves in time, δ_t moves along at velocity $a^2 - 3b^2$. WLOG, choose $\left|\alpha e^{qx+q^3 t}\right| = |\alpha| \left|e^{qx+q^3 t}\right|$. According to Proposition 3.6,

$$|\alpha| \left|e^{qx+q^3 t}\right| = |\alpha| e^{a(x+(a^2-3b^2)t)}.$$

Now, if we shift this magnitude in time by some amount, say by Δt, then

$$|\alpha| e^{a(x+(a^2-3b^2)(t+\Delta t)} = |\alpha| e^{a(x+(a^2-3b^2)\Delta t)+(a^2-3b^2)t}.$$

Thus shifting x by the amount of time multiplied by the velocity, as desired. Thus the choice of δ is t-independent. ■

4.2 Application to 2-Soliton Scattering ★

The previous theorem, reduced to $n = 2$, allows for a complete description of the 2-soliton interaction and its associated phase shift. First, we will set up some notation.

Let

$$\phi_i := \phi(q_i, \alpha_i, \beta_i) = \alpha_i e^{q_i x + q_i^3 t} + \beta_i e^{-q_i x - q_i^3 t}$$

index the (two) functions in $\mathcal{F}$, and let $q_i = a_i + b_i \boldsymbol{i}$. To remain consistent, I will call the soliton with lowest velocity ϕ_1, and the other ϕ_2 (as defined in Definition (3.5)). This way, the solitons are spatially ordered from left to right for $t >> 0$. It is assumed that they do not have the same velocity, since in this case, there is no collision.

Definition 4.1. Let Q_i^+ be the unique monic differential operator whose kernel contains $\phi_i - \beta_i e^{-q_i x - q_i^3 t} = \alpha_i e^{q_i x + q_i^3 t}$ and let Q_i^- be the unique monic differential operator whose kernel contains $\phi_i - \alpha_i e^{q_i x + q_i^3 t} = \beta_i e^{-q_i x - q_i^3 t}$. Define $\phi_i^\pm = Q_i^\pm(\mathcal{F}\backslash\{\phi_i\})$, where $Q_i^\pm$ is applied element-wise to $\mathcal{F}\backslash\{\phi_i\}$.

Working this out for $n = 2$, we have

$$\begin{aligned}
\phi_1^- &= \phi\left(q_2, \beta_1 q_1 \beta_1^{-1}\alpha_2 + \alpha_2 q_2, \beta_1 q_1 \beta_1^{-1}\beta_2 - \beta_2 q_2\right)\\
\phi_2^- &= \phi\left(q_1, \beta_2 q_2 \beta_2^{-1}\alpha_1 + \alpha_1 q_1, \beta_2 q_2 \beta_2^{-1}\beta_1 - \beta_1 q_1\right)\\
\phi_1^+ &= \phi\left(q_2, -\alpha_1 q_1 \alpha_1^{-1}\alpha_2 + \alpha_2 q_2, -\alpha_1 q_1 \alpha_1^{-1}\beta_2 - \beta_2 q_2\right)\\
\phi_2^+ &= \phi\left(q_1, -\alpha_2 q_2 \alpha_2^{-1}\alpha_1 + \alpha_1 q_1, -\alpha_2 q_2 \alpha_2^{-1}\beta_1 - \beta_1 q_1\right).
\end{aligned}$$

Corollary 4.3.1. *Let $u_+(x,t)$ and $u_-(x,t)$ be KdV 1-soliton solutions:*

$$u_+(x,t) = \Omega(Q_+(\phi_2)) \quad Q_+ = \partial - \alpha_1 q_1 \alpha_1^{-1},$$

$$u_-(x,t) = \Omega(Q_-(\phi_1)) \quad Q_- = \partial - \beta_2 q_2 \beta_2^{-1}.$$

For sufficiently large values of t, the solution $u(x,t) = \Omega(\{\phi_1, \phi_2\})$ will look like their sum. That is, for $t >> 0$,

$$u(x,t) \approx u_+(x,t) + u_-(x,t).$$

Proof. Let $\epsilon > 0$ be given. As given in Theorem 4.3, we have a distance δ_1 so that for any time t, if we choose x more that δ_1 away from $c_{\phi_1}(t)$ on the right, the solution $u(x,t)$ is within ϵ of $u_+(x,t)$ at every point. Similarly, there exists a δ_2 so that far enough away from $c_{\phi_2}(t)$, $u(x,t)$ looks like $u_-(x,t)$ on the left.

Since the centers are moving at different velocities, t can be chosen so that they are arbitrarily far apart. Note that u_+ and u_- are 1-soliton solutions with the same velocities as $\Omega(\phi_1)$ and $\Omega(\phi_2)$, respectively. So for some time t', $|c_{\phi_2}(t') - c_{\phi_1}(t')| > \max\{\delta_1, \delta_2\}$. Thus on the right, u is well approximated by u_+ and on the left, u is well approximated by u_-. Since for every calue of x, u looks like either u_- or u_+ while the other is negligibly small, we have that $u \approx u_+(x,t) + u_-(x,t)$. ■

I will use the forms from Definition 4.1 to try and relate the linear trajectories of the quaternionic 2-soliton interaction. We are assuming that $a_i \neq 0$, so that the soliton is localized.

Lets focus first on ϕ_1. To reiterate, ϕ_1^+ is the function corresponding to the 1-soliton on the right of ϕ_1, while ϕ_1^- corresponds to the left of ϕ_1. Therefore after impact, the center has trajectory:

$$c_{\phi_1^-}(t) = \frac{\ln\left|(\beta_1 q_1 \beta_1^{-1}\alpha_2 + \alpha_2 q_2)^{-1}(\beta_1 q_1 \beta_1^{-1}\beta_2 - \beta_2 q_2)\right|}{2a_2} - v_2 t$$

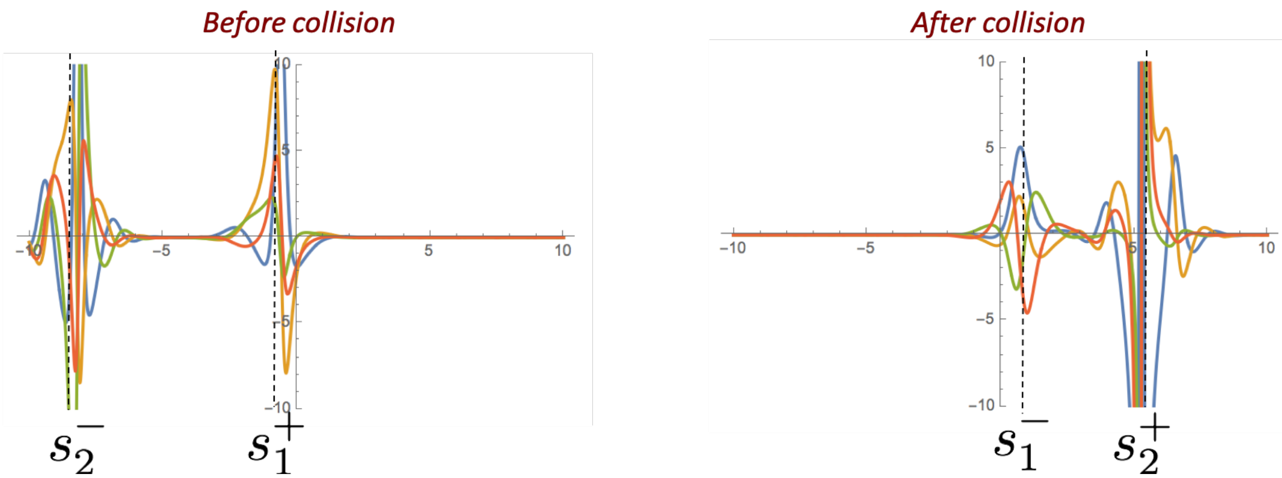

Figure 7: A example quaternionic 2-soliton solution to the ncKdV equation depicted before collision and after collision with centers labeled according to Definition (4.1).

and before impact:

$$c_{\phi_1^+}(t) = \frac{\ln|(-\alpha_1 q_1 \alpha_1^{-1}\alpha_2 + \alpha_2 q_2)^{-1}(-\alpha_1 q_1 \alpha_1^{-1}\beta_2 - \beta_2 q_2)|}{2a_2} - v_2 t$$

So their linear trajectories differ by [10]

$$\begin{aligned}
c_{\phi_1^-}(t) - c_{\phi_1^+}(t) &= \frac{1}{2a_2}\left[\ln\left(\frac{|\beta_1 q_1 \beta_1^{-1}\beta_2 - \beta_2 q_2|}{|\beta_1 q_1 \beta_1^{-1}\alpha_2 + \alpha_2 q_2|}\right) - \ln\left(\frac{|-\alpha_1 q_1 \alpha_1^{-1}\beta_2 - \beta_2 q_2|}{|-\alpha_1 q_1 \alpha_1^{-1}\alpha_2 + \alpha_2 q_2|}\right)\right] \\
&= \frac{1}{2a_2}\left[\ln\left(\frac{|(\beta_1 q_1 \beta_1^{-1}\beta_2 - \beta_2 q_2)(\alpha_1 q_1 \alpha_1^{-1}\alpha_2 - \alpha_2 q_2)|}{|(\beta_1 q_1 \beta_1^{-1}\alpha_2 + \alpha_2 q_2)(\alpha_1 q_1 \alpha_1^{-1}\beta_2 + \beta_2 q_2)|}\right)\right] \\
&= \frac{1}{2a_2}\left[\ln\left(\frac{|(\beta_1 q_1 \beta_1^{-1} - \beta_2 q_2 \beta_2^{-1})(\alpha_1 q_1 \alpha_1^{-1} - \alpha_2 q_2 \alpha_2^{-1})|}{|(\beta_1 q_1 \beta_1^{-1} + \alpha_2 q_2 \alpha_2^{-1})(\alpha_1 q_1 \alpha_1^{-1} + \beta_2 q_2 \beta_2^{-1})|}\right)\right]
\end{aligned}$$

Note that in the real case, where $\alpha = 1$, we have the usual

$$c_{\phi_1^-}(t) - c_{\phi_1^+}(t) = \frac{1}{a_2}\ln\left|\frac{q_1 - q_2}{q_1 + q_2}\right|.$$

All of this work is nearly identical for ϕ_2, but since ϕ_2 is on the right for $t << 0$, $c_{\phi_2^-}(t)$ actually describes the soliton associated with ϕ_1 *before* collision and $c_{\phi_2^+}(t)$ describes the soliton after collision. Therefore the difference is

$$c_{\phi_2^+}(t) - c_{\phi_2^-}(t) = -\frac{\ln\epsilon}{2a_1}.$$

where

$$\epsilon = \frac{|(\beta_1 q_1 \beta_1^{-1} - \beta_2 q_2 \beta_2^{-1})(\alpha_1 q_1 \alpha_1^{-1} - \alpha_2 q_2 \alpha_2^{-1})|}{|(\beta_1 q_1 \beta_1^{-1} + \alpha_2 q_2 \alpha_2^{-1})(\alpha_1 q_1 \alpha_1^{-1} + \beta_2 q_2 \beta_2^{-1})|}$$

Now we can describe the lines dictating the linear trajectory of the soliton centers.

[10]Note: $\alpha, \beta \in \mathbb{H} \Rightarrow |\alpha|/|\beta| = |\alpha\beta^{-1}| = |\beta^{-1}\alpha|$

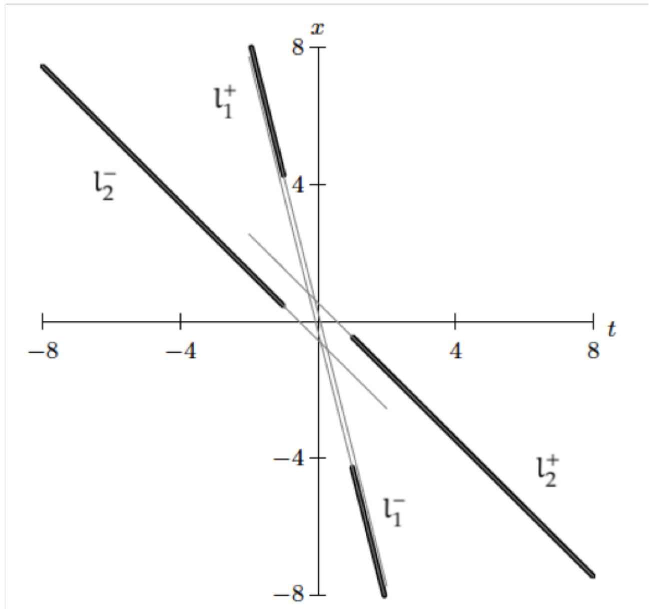

Figure 8: An example graph of the lines defined in Definition (4.2).

Definition 4.2. Let $u_2(x,t)$ be a 2-soliton solution to the KdV equation. Additionally, to shorten notation, let

$$\xi_2 = \ln \frac{|\beta_1 q_1 \beta_1^{-1} \alpha_2 + \alpha_2 q_2|}{|\beta_1 q_1 \beta_1^{-1} \beta_2 - \beta_2 q_2|} \quad \text{and} \quad \xi_1 = \ln \frac{|\alpha_2 q_2 \alpha_2^{-1} \beta_1 - \beta_1 q_1|}{|\alpha_2 q_2 \alpha_2^{-1} \alpha_1 - \alpha_1 q_1|}.$$

We define the following lines:

$$l_1^- : x = v_2 t + \frac{\xi_2 + \ln \epsilon}{2a_2} \qquad l_2^- : x = v_1 t + \frac{\xi_1}{2a_1}$$

$$l_1^+ : x = v_2 t + \frac{\xi_2}{2a_2} \qquad l_2^+ : x = v_1 t + \frac{\xi_1 - \ln \epsilon}{2a_1}$$

Remark. *My notation differs from [3] here. The lines with negative superscripts in BKY are solitons to the left of the collision. Since quaternionic solitons can move both left and right, the frame of reference has been changed. For example, l_1^- shows the path of the soliton left of ϕ_1.*

Proposition 4.4. *As $t \to \pm\infty$, $\Omega(\{\phi_1, \phi_2\}) \approx s_2^{\pm} + s_1^{\mp}$, which have asymptotic linear trajectories given by the lines $l_2^{\pm}$ and $l_1^{\mp}$.*

4.3 Application to Rational Solutions

At the end of Section 3.2, it was stated without proof that one only needs to consider polynomials in $\mathcal{B}$ of odd order. With Theorem 3.4, this can be easily explained.

Let $\Phi \in \mathcal{B}$ be of even order m. Then as detailed in Definition (3.4),

$$\mathcal{F}(\Phi) = \{\Phi, \Phi_{xx}, \Phi_{xxxx}, \dots, \Phi_m\}$$

Now, the m-th derivative of a polynomial Φ of order m is a constant, so $\Phi_m = c \in \mathbb{H}$. Since this is of the form of a trivial eigenfunction, we can apply Theorem (4.2) using $Q = \partial$ to conclude that

$$\Omega(\mathcal{F}(\Phi)) = \Omega\left(\partial(\widehat{\mathcal{F}})\right).$$

where ∂ is applied to each element in the subset $\widehat{\mathcal{F}} = \mathcal{F}(\Phi)\backslash\Phi_m$:

$$\partial\widehat{\mathcal{F}} = \{\Phi_x, \Phi_{xxx}, \ldots, \Phi_{m-1}\} = \mathcal{F}(\Phi_x).$$

Since

$$\Phi_x = \frac{\partial}{\partial x}\sum_{i=0}^{m}\Delta_i\alpha_i = \sum_{i=0}^{m} i\Delta_{i-1}\alpha_i,$$

is a function of odd order $m-1$, we've found that for every even order choice of Φ in $\mathcal{F}(\Phi)$, we could always use the smaller odd ordered Φ_x in $\mathcal{F}(\Phi_x)$ corresponding to the same solution u. Moreover, it is also sufficient to only consider the even indexed Δ_i's for the lower terms in Φ (see Serna, Albert, master's thesis, College of Charleston, 2019).

5 Conclusion

The non-commutative KdV equation boasts a rich mathematical structure left mostly unexplored by investigations of the past. Earlier work in this direction can be found in [9], [11], or [13], but this paper contributes a thorough discussion of quaternionic solution behavior in comparison to commutative results. Results for quaternion soliton theory finds application in physical systems where noncommutativity plays a large role, such as in quaternionic quantum mechanics [11] [25].

Like in the case for *complexitons* [12], quaternionic 1-solitons can travel in either direction, and consist of envelope and periodic parts. However, quaternionic solitons for the KdV equation differ from the familiar real and complex settings in two striking ways when viewed as planar waves:

1. Although a quaternion soliton $u(x,t) \to 0$ as $x \to \pm\infty$ like normal, the two sides of the soliton can be different from one another. In particular, the sides are rotations of one another.
2. The envelope and periodic parts have distinct velocities.

These differences give marked changes in quaternionic 2-soliton interactions. Although the envelope shape and speeds of the solitons are preserved in all cases, the shift in position or phase shift is dependent on the quaternionic coefficients in addition to their incoming velocities.

6 Future Work

Currently, the generalization of the phase shift is real-valued, so that it still corresponds to a physical shift along the spatial axis. One interesting direction would be to derive a quaternionic-valued phase shift, whose modulus corresponds to its current value, and its direction corresponds

[11]Problems with using quaternions in such systems mostly lie with a more restricted concept of quaternion analyticity.

to the rotation that a soliton goes through upon collison. Another interesting direction would be extend this theory to non-associative algebras, such as the octonions. Recently, after a presentation of the material in this talk, we discovered an interesting correspondence inspired by [26] between the soliton solution structure and particular Jordan algebras that hint at the possibility for an octonion extension. Another interesting direction would be to see what the properties of a noncommutative Sato Grassmannian, parameterized by solutions to the nonabelian KP hierarchy. Lastly, I hope to extend the theory of bilinear KdV equations and the ensuing "direct form" found by Hirota [15] to the noncommutative case.

7 Acknowledgements

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Non-commutative n-KdV Hierarchy and the ncKdV Equation

The quaternionic (or more generally non-commutative) KP hierarchy is basically identical to the commutative KP hierarchy, as seen in [14]. Letting

$$\mathcal{L} = \partial + \sum_{i=1}^{\infty} u_i \partial^{-l}$$

be a pseudo-differential operator where $u_i \in \mathbb{H}[[x]]$ and $\partial = \frac{\partial}{\partial x}$, the quaternionic KP hierarchy is defined by the flows

$$\frac{\partial}{\partial t_m}\mathcal{L} = [(\mathcal{L}^m)_+, \mathcal{L}]$$

for $m \in \mathbb{N}$, where $\mathcal{L}_+$ denotes the differential part of $\mathcal{L}$ with nonnegative powers of ∂.

If one restricts the space of operators $\mathcal{L}$ to only those where $\mathcal{L}^m = L$ is an ordinary differential operator,

$$\frac{\partial}{\partial t_m} L = [(L^{m/n})_+, L]$$

defines the m^{th} equation of the n-KdV hierarchy with solutions L. The third equation of the non-commutative 2-KdV hierarchy,

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}u_x u + \frac{3}{4}u u_x, \tag{14}$$

is the non-commutative KdV equation or ncKdV equation. Note that we have made $t = t_3$ for simplicity. The ncKdV equation remains solvable by many traditional routes (subject to slight generalization). In this paper, we investigate the gauge-like transformations through Darboux transformations, although there may be an interesting connection to Hirota's direct method using τ-functions using quasideterminants.

Introduction: The KdV Equation

Although there are an infinite number of solutions to the KdV Equation

$$u_t = \frac{3}{2}u u_x + \frac{1}{4}u_{xxx}, \tag{15}$$

there are only some that satisfy the "soliton" behavior studied in soliton theory. These are called pure n-soliton solutions, named for their n number of humps.

Definition 7.1. [12] We say that the function $u(x,t)$ is a **pure n-soliton solution** of the KdV Equation if all of the following apply:

- It is a solution of the KdV Equation.
- It is continuous for all x and t.
- $\lim_{x \to \pm\infty} u(x,t) = 0$.

[12] noted to be ambiguous across mathematics.

- It can be written in the rational-exponential form:

$$u(x,t) = \frac{\sum_{i=1}^{m} c_i e^{a_i x + b_i t}}{\sum_{j=1}^{m} C_j e^{A_j x + B_j t}}$$

for some positive integers m and n and real numbers a_i, b_i, c_i, A_j, B_j, and C_j

- For sufficiently large values of $|t|$ the graph of $y = u(x,t)$ has n local maxima.

A surprising fact is that all n-soliton solutions to the KdV equation are in the form

$$u(x,t) = 2\partial_x^2 \ln(\tau(x,t)) = \frac{2\tau\tau_{xx} - 2\tau_x^2}{\tau^2} \tag{16}$$

for some mystery function $\tau(x,t)$ of the form $\sum_{i=1}^{n} c_i e^{a_i x + b_i t}$. It should be surprising that all solutions can essentially be described by such a simple function, and renders the τ functions more "fundamental" in this sense. It turns out that

$$\tau(x,t) = \det \mathrm{Wr}(\phi_1, \ldots, \phi_n),$$

where ϕ_i denote the functions in the KdV-Darboux kernel $\mathcal{F}$. The questions I'm pursuing are

1. Prove and figure out how τ-functions solve KdV.
2. Determine properties of τ-functions.
3. Generalize to KP
4. Are there equivalent quaternion-valued τ-functions?

KP Equation & Bilinear KP Equation

The KP Equation is significant because it "contains" KdV inside of it, as shown below.

Definition 7.2. The **KP Equation** is the nonlinear partial differential equation for a function $u(x,y,t)$ that can be written as

$$\begin{aligned} u_{yy} &= \frac{4}{3}u_{xt} - 2u_x^2 - 2uu_{xx} - \frac{1}{3}u_{xxxx} \\ &= \frac{4}{3}\frac{\partial}{\partial x}\underbrace{\left(u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}\right)}_{\text{KdV}} \end{aligned} \tag{17}$$

Remark. *All functions $u(x,t)$ that solve the KdV equation also solve the KP equation since if $u(x,t)$ solves KdV, the right hand side is 0, and since $u(x,t)$ independent of y, the left hand is also zero.*

Therefore KdV solutions are just solutions to KP that happen to be independent of y. The KP Equation is practically the KdV Equation with two spatial dimensions, and its solutions $u(x,y,t)$ can be visualized as a surface at each time t. Just as in the KdV case, there is a corresponding τ-function that "magically" produces infinite amounts of solutions.

Bilinear KP Equation

If an equation admits soliton solutions, it has become natural to search for a bilinear counterpart[13]. This is because their solutions end up following the linear superposition principle, even though the bilinear KdV equation is nonlinear.

Definition 7.3. Let $\tau(x,y,t)$ be a function of $x,y,$ and t. We say it is a τ-**function** for the KP Equation if it satisfies the **Bilinear KP Equation**:

$$-3\tau_y^2 + 3\tau_{xx}^2 + 3\tau\tau_{yy} + 4\tau_t\tau_x - 4\tau\tau_{xt} - 4\tau_x\tau_{xxx} + \tau\tau_{xxxx} = 0. \tag{18}$$

This is related to the KP Equation in the following way:

Theorem 7.1. *Any nonzero τ-function can be turned into a solution of the KP Equation (17) using*

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\tau(x,y,t).$$

The general definition for a bilinear differential equation is written in the terms of *Hirota Derivatives*[14], but it might help to see that every term on the left side is a product of exactly two copies of τ or its derivatives. This allows for the following theorem:

Theorem 7.2 (Gauge transformation)**.** *$\tau(x,y,t)$ is a τ-function for the KP Equation if and only if*

$$\bar{\tau}(x,y,t) = \lambda e^{\alpha x+\beta y+\gamma t}\tau(x,y,t)$$

is also a solution for any constants $\lambda,\alpha,\beta,\gamma$.
Moreover, if they are, then they correspond to the same solution $u(x,y,t)$ of the KP Equation (17);

$$u = 2\partial_x^2\ln\tau = 2\partial_x^2\ln\bar{\tau}.$$

Proof. Let F be the left side of the Bilinear KP Equation,

$$F(f(x,y,t)) = -3f_y^2 + 3f_{xx}^2 + 3ff_{yy} + 4f_tf_x - 4ff_{xt} - 4f_xf_{xxx} + ff_{xxxx}$$

so that $F(\tau) = 0$ and let $\bar{\tau} = \lambda e^{\alpha x+\beta y+\gamma t}\tau$. One could confirm that $F(\bar{\tau}) = \lambda^2 e^{2(\alpha x+\beta y+\gamma t)}F(\tau)$. Therefore $F(\bar{\tau}) = 0$. ∎

Remark. *Linear combinations of τ-functions are not always τ-functions - we have only shown that scalar multiples of real numbers and that weird functions are still τ-functions. This is addressed in the next section.*

[13]there is also a corresponding bilinear KdV equation.
[14]more information in [15]

Definition 7.4 (Nicely Weighted Functions)**.** Let us say that $\tau(x, y, t)$ is a nicely weighted function[15] if it satisfies the two linear equations

$$\tau_{xx} = \tau_y \qquad \tau_{xxx} = \tau_t \tag{19}$$

Theorem 7.3. *Every nicely weighted function is a τ-function.*

Proof. Simply replace every $\tau_y = \tau_{xx}$ and every $\tau_t = \tau_{xxx}$ in the Bilinear KP Equation (18). This will cancel each term with its neighbor. ∎

Similar to the "vacuum eigenfunction" e^{kx+k^3t} constructed to satisfy $\phi_{xxx} = \phi_t$, there is a "vacuum eigenfunction" $e^{xz+yz^2+tz^3}$ that obviously solves the two linear equations to be a nicely weighted function for any real number z (and is therefore a τ-function). However, note that we can do better than simply multiplying by a constant since it is nicely weighted: we can take linear combinations of this function, and it will still be nicely weighted *and* may lead to different solutions $u(x, y, t)$ to KP. Even further, since differentiation with respect to the free parameter z commutes with differentiation of x, y, and t, we can make more nicely weighted functions:

Definition 7.5. For any number λ and any nonnegative integer n, let $\varphi_\lambda^{(n)}(x, y, t)$ denote the nicely weighted function

$$\varphi_\lambda^{(n)}(x, y, t) = \frac{\partial^n}{\partial z^n} e^{zx+z^2y+z^3t}\bigg|_{z=\lambda}.$$

So, for example, $2\varphi_3^{(0)} + 5\varphi_\pi^{(67)} - 22\varphi_{-0.2}^{(2)}$ is another nicely weighted function and therefore a τ-function.

There is one more way to combine nicely weighted functions, which is more interesting because it can lead to τ-functions that are *not* nicely weighted.

Theorem 7.4 (Wronskians of Nicely Weighted Functions are also τ-functions)**.** *Pick N linearly independent functions $\varphi_1(x, y, t), \ldots, \varphi_N(x, y, t)$ each of which is a nicely weighted function. Then their Wronskian*

$$\tau = Wr(\varphi_1, \ldots, \varphi_N) = \det \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_N \\ \frac{\mathrm{d}}{\mathrm{d}x}\varphi_1 & \frac{\mathrm{d}}{\mathrm{d}x}\varphi_2 & \vdots & \frac{\mathrm{d}}{\mathrm{d}x}\varphi_N \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\mathrm{d}^{N-1}}{\mathrm{d}^{N-1}x}\varphi_1 & \frac{\mathrm{d}^{N-1}}{\mathrm{d}^{N-1}x}\varphi_2 & \cdots & \frac{\mathrm{d}^{N-1}}{\mathrm{d}^{N-1}x}\varphi_N \end{pmatrix}$$

solves the Bilinear KP Equation (18) and is therefore a τ-function, not necessarily nicely weighted.

Proof. There is really nice proof in [4] starting on page 189, and it may be worthwhile to revisit at a later time when thinking more about quaternions. ∎

[15] referred to as "adjoint eigenvector" in literature.

Summary

The KP Equation is like the KdV equation but with 2-spatial dimensions - its solutions are traveling waves on a surface. All solutions to the KdV equation are also solutions to the KP equations. Likewise, all solutions $u(x, y, t)$ to the KP Equation that happen to be y independent (i.e. evaluated at some y_0 so that $u(x, y_0, t)$ are solutions to the KdV equation.)

The KP Equation has a bilinear counterpart called the Bilinear KP Equation (18). Every solution of this equation is called a τ-function and describes solutions to the KP equation via $u = 2(\ln \tau)_{xx}$. But what do general τ functions look like? Its easier to talk about a subset of τ-functions called nicely weighted functions, which satisfy $f_{xxx} = f_t$ and $f_{xx} = f_y$. A principal example of this is the nicely weighted function $e^{zx+z^2y+z^3t}$. Since these are τ-functions, they solve the Bilinear KP Equation and can be transformed to solutions to KP. There are two ways that these nicely weighted function can produce more τ-functions.

1. They can be differentiated/evaluated with respect to z and linearly combined and to make new nicely weighted functions.

2. The Wronskian of a collection of nicely weighted functions is a τ-function, not necessarily nicely weighted.

τ and the 1-soliton (nonessential)

The 1-soliton solutions of the KP Equation comes from letting $\tau(x, y, t)$ be the sum of nicely weighted functions:

$$\tau(x, y, t) = \varphi^{(0)}_{\lambda_1}(x, y, t) + \gamma\varphi^{(0)}_{\lambda_2}(x, y, t) \qquad \gamma > 0, \lambda_1 \neq \lambda_2$$

This produces the KP solution

$$u(x, y, t) = \frac{2(\lambda_1 - \lambda_2)^2 e^{(\lambda_1+\lambda_2)x+(\lambda_1^2+\lambda_2^2)y+(\lambda_1^3+\lambda_2^3)t}}{(e^{\lambda_1 x+\lambda_1^2 y+\lambda_1^3 t} + e^{\lambda_2 x+\lambda_2^2 y+\lambda_2^3 t})^2}$$

for $\gamma = 1$. This solution looks like a straight line wave front with slope $-1/(\lambda_1 + \lambda_2)$ in the xy-plane and height $\frac{1}{2}(\lambda_1 - \lambda_2)^2$. In the event that $\lambda_1 = -\lambda_2$, this reduces to the KdV soliton form.

n-soliton solutions can be produced by combining several 1-solitons solutions by using the Wronskian determinant method above.

The Grassmann Cone $\Gamma_{2,4}$ & Bilinear KP Equation

The Grassmann Cone $\Gamma_{2,4}$

Wedge Products & Notation

Let V and W be vector spaces of dimension 4 and 6, respectively.

Let the basis vectors in V be written $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ so that any $\Phi \in V$ can be written

$$\Phi = \sum_{i=1}^{4} c_i\phi_i$$

uniquely for scalar coefficients c_i.

Similarly, let $\{\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}\}$ be the basis vectors for W[16] so that any $\Omega \in W$ can be written uniquely as

$$\Omega = \sum_{1\leq i<j\leq 4} c_{ij}\omega_{ij}.$$

Now, we will define the wedge product $\wedge : V \to W$.

Definition 7.6. The wedge product is characterized by the following properties:

- $\phi_i \wedge \phi_j = \omega_{ij}$ if $i < j$.
- $\Phi_1 \wedge (a\Phi_2 + b\Phi_3) = a(\Phi_1 \wedge \Phi_2) + b(\Phi_1 \wedge \Phi_3)$ if $\Phi_i \in V$ and $a, b \in \mathbb{R}$.
- $\Phi_1 \wedge \Phi_2 = -\Phi_2 \wedge \Phi_1$.

Corollary 7.4.1.

- *Any Φ wedged with itself is 0.*
- *If $\Phi_1 = \sum_{i=1}^4 a_i\phi_i$ and $\Phi_2 = \sum_{i=1}^4 b_i\phi_i$, then the coefficient of ω_{ij} in*

$$\Phi_1 \wedge \Phi_2 = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} = a_ib_j - a_jb_i.$$

Note that only some elements in W can be written as the wedge product of two elements in V: these are the *decomposable* ones. For example, $-3\omega_{12} - 3\omega_{13} + \omega_{14} + 3\omega_{23} + 2\omega_{24} + 3\omega_{34} = (\phi_1 + 2\phi_2 + 3\phi_3) \wedge (\phi_1 - \phi_2 + \phi_3)$ but $\omega_{12} + \omega_{34}$ cannot be "factored" in this way and so is *indecomposable.*

Grassmann Cone & Plücker Relation

The decomposable elements in W collectively make up the Grassmann cone.

Definition 7.7. Let

$$\Gamma_{2,4} = \{\Omega \in W | \Omega = \Phi_1 \wedge \Phi_2 \text{ for some vectors } \Phi_i \in V\}$$

be the set of *decomposable* elements in W. We call it the *Grassmann Cone.*

But how do we find these elements?

Theorem 7.5. *The vector $\Omega = \sum_{1\leq i<j\leq 4} c_{ij}\omega_{ij}$ is in $\Gamma_{2,4}$ if and only if the coefficients satisfy the Plücker relation:*

$$c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}.$$

Proof. See page 201 in GOST. ∎

So, the Grassmann cone is a algebro-geometric object created by the subset of points in W whose coordinates satisfy a algebraic equation. We can work out (later) the geometric properties of the Grassmann cone.

[16] seems strange, but roll with it, the notation makes sense in the context of wedge product properties

- **Connectivity:** The Grassmann cone is *connected* in the sense that you can "travel" to any other point on the Grassmann cone without leaving it.

- **Dimension:** Since a curve has a tangent line, it is dimension 1, and a surface has a tangent plane, it has dimension 2. Similarly, the Grassmann has a tangent space, and this is its dimension.

- **Cone:** If Ω is decomposable, the so is $\lambda\Omega$ for any number λ, which means the Grassmann cone is made up of lines through the origin (when $\lambda = 0$). Slices of the cone are called *Grassmannians*, which are somehow of more fundamental importance than the Grassmann cone.

GEOMETRY OF PDE'S AND BILINEAR KP

By letting V be a four-dimensional space spanned by nicely weighted functions ϕ_i $(1 \leq i \leq 4)$, with the wedge product defined in terms of Wronskians, the Plücker relation is equivalent to the Bilinear KP Equation. That is, if you let $\omega_{ij} = \mathrm{Wr}(\phi_i, \phi_j)$ so that the Wronskians of the basis elements of V make a new 6 dimensional space W with basis $\{\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}\}$, and let $\Phi_1 \wedge \Phi_2 = \mathrm{Wr}(\Phi_1, \Phi_2)$.

Then we can ask if a general function $\tau \in W$ is decomposable and thus a τ-function. As shown earlier, this is equivalent to asking if its coordinates with respect to basis in W satisfy the Plücker relation. Whats not obvious is that the Plücker relation has identical solutions to the Bilinear KP Equation.

Theorem 7.6. *Let $\phi_i(x, y, t)(i = 1, 2, 3, 4)$ be linearly independent nicely weighted functions and let W be the 6-dimensional vector space of functions spanned by*

$$\omega_{ij}(x, y, t) = Wr(\phi_i, \phi_j) \qquad 1 \leq i < j \leq 4$$

where differentiation is taken with respect to x. Then a general element of W

$$\tau(x, y, t) = \sum_{1 \leq i < j \leq 4} c_{ij}\omega_{ij}(x, y, t)$$

is a solution to the Bilinear KP Equation if and only if the coefficients c_{ij} satisfy the algebraic Plücker relation (10.4)

Proof. Page (206) of GOST ∎

The significance of Theorem 7.6 is that a small part of the solution set to Bilinear KP is given the geometry of the Grassmann Cone $\Gamma_{2,4}$.

For every algebrogeometric object, there is a corresponding partial differential equation whose solutions, which are linear combinations of functions, is exactly those whose coefficients satisfy the desired algebraic equation of your geometric object. Note that many different partial differential equations can be arbitrarily constructed so that determining if a linear combination of functions is a solution is equivalent to an algebraic equation corresponding to a familiar geometric object. For a (trivial) example, its not surprising that

$$g(u, v, w, x, y, z) = c_{12}u + c_{13}w + c_{24}x + c_{14}y + c_{23}z + c_{34}z$$

only solves the partial differential equation

$$g_u g_v - g_w g_x + g_y g_z = 0$$

if the coefficients of g satisfy the Plücker relation above because, for instance, g_u is just a tricky way to write c_{12}. By isolating coefficients using differentiation and evaluation[17], we can show that the Bilinear KP can be generated in the same way from the Plücker relation.

So Theorem 7.6 selects coordinates of points in a Grassmann Cone for a wide variety of different linear combinations of solutions.

Pseudo-Differential Operators & the KP Hierarchy

Pseudo-Differential Operators

Earlier, higher order KdV-like equations were constructed by assuming that the differential operator M in the Lax pair $\cdot L = [M, L]$ had order larger than 3. The point of pseudo-differential operators is to allow inverses to differential operators, which will allow a more general definition of KdV hierarchies. Keep in mind that these no longer make sense as operators – they cannot act on functions since there are multiple functions with the same derivative.

Definitions

Definition 7.8. A *pseudo-differential operator* (Ψ DO) is an infinite sum of the form

$$\mathcal{L} = \sum_{i=-\infty}^{N} \alpha_i(x)\partial^i.$$

We say that $\mathcal{L}$ has order N. Note also that $\mathcal{L}_+$ and $\mathcal{L}_-$ refer to the truncated terms

$$\mathcal{L}_+ = \sum_{i=0}^{N} \alpha_i(x)\partial^i \quad \text{and} \quad \mathcal{L}_- = \sum_{i=-\infty}^{-1} \alpha_i(x)\partial^i$$

so that $\mathcal{L}_+$ is actually an ordinary differential operator and $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$.

These can be added together by collecting functions with the same power, but multiplication is more difficult. Since we want ∂^{-1} to be the multiplicative inverse to ∂.[18] We must have

$$\partial^{-1} \circ f = f\partial^{-1} - f'\partial^{-2} + f''\partial^{-3} - \cdots = \sum_{i=0}^{\infty} (-1)^i f^{(i)}(x)\partial^{-1-i}.$$

In general, multiplication can be defined as

$$f(x)\partial^i \circ g(x)\partial^j = \sum_{n=0}^{\infty} \binom{i}{n} f(x) g^{(n)}(x)\partial^{i+j-n}.$$

Where the definition of

$$\binom{i}{n} = \begin{cases} \frac{i(i-1)(i-2)\cdots(i-n+1)}{n!} & n > 0, \\ 1 & n = 0, \end{cases}$$

has been expanded to include $n > i$.

Now, we can use the following theorem.

[17] details on page 214, I have a few questions about the choice of basis for V and example 10.12.

[18] so that $(\partial^{-1} \circ \partial \circ f = f)$.

Theorem 7.7. *Let L be an ODO. Then there exists a multiplicative inverse ΨDO L^{-1} and a ΨDO $L^{1/n}$ with the property that $(L^{1/n})^n = L$.*

n-KdV Hierarchy

Theorem 7.8. *Let L be the monic ordinary differential operator*

$$L = \partial^n + \sum_{i=0}^{n-2} u_i(x,t)\partial^i,$$

and define $M = (L^{k/n})_+$. Then $\hat{L} = [M, L]$ is a soliton equation we call the k^{th} equation of the n-KdV Hierarchy.

Proof. Earlier in the book, we assumed M was a differential operator of a different order, and then canceled out the necessary terms in $\hat{L} = [M, L]$ so that the equation makes sense. The proof of this similarly just verifies that both sides of the Lax form are of order $n - 2$. ■

Although this is a closed form for the KdV-type equations, it might be arguably harder to find since we'd need to find $L^{k/n}$.

KP Hierarchy

The KP Hierarchy contains all of the equations in the n-KdV Hierarchies, the KP Equation, and others. It is strange at first because it is written in terms of infinitely many variables and pseudo-differential operators.

Definition 7.9. Define $\mathbf{t} = (t_1, t_2, t_2, \dots)$ to be the collection of infinitely many variables t_i with index i running over the positive integers. Its convenient to let the first three be the variables already introduced for the KP Equation so that

$$x = t_1 \quad y = t_2 \quad t = t_3.$$

We say that a pseudo-differential operator

$$\mathcal{L} = \partial + \sum_{i=1}^{\infty} a_i(\mathbf{t})\partial^{-i}$$

is a solution to the KP Hierarchy if it solves the Lax Equations

$$\frac{\partial}{\partial t_i}\mathcal{L} = [(\mathcal{L}^i)_+, \mathcal{L}]$$

for every i.

There are many connections between the KP Hierarchy and previously encountered soliton equations. For example:

Theorem 7.9. *If* $\mathcal{L} = \partial + a_1(\mathbf{t})\partial^{-1} + \dots$ *is a solution to the KP Hierarchy then the function* $u(x, y, t, t_4, t_5, t_6, \dots) = 2a_1(x, y, t, t_4, t_5, t_6, \dots)$ *is a solution to the KP Equation (17) for any fixed values of* t_i *for* $i \geq 4$.

Note also that the KdV Hierarchy is a subset of the KP Hierarchy. The $\mathcal{L}$ that satisfies the KP Hierarchy and is also a ordinary differential operator $\mathcal{L}^n$ for some n the KP Hierarchy reduces to the KdV Hierarchy for $\mathcal{L}^n$.

Lemma 7.10. *If* $\mathcal{L}^n$ *is an ordinary differential operator, then* $\frac{\partial}{\partial t_n}\mathcal{L} = 0$.

Theorem 7.11. *If* $\mathcal{L}$ *is a solution to the KP Hierarchy and* $L = \mathcal{L}^n$ *is an ordinary differential operator, then*

$$\frac{\partial}{\partial t_k}\mathcal{L} = [(L^{k/n})_+, L]$$

so that L *is a solution of the* k^{th} *equation of the* n*-KdV Hierarchy with respect to the variable* t_k.

Even after all this, we have yet to discuss the way to make $\mathcal{L}$, and by extension, solutions to every soliton equation. The construction follows a similar idea as Theorem 8.3 in GOST.

Theorem 7.12. *Let* $\phi_1(\mathbf{t})$, $\phi_2(\mathbf{t})$, $\dots$, $\phi_n(\mathbf{t})$ *be functions with a nonzero Wronskian each of which satisfies the conditions*

$$\frac{\partial}{\partial t_1^i}\phi_j(\mathbf{t}) = \frac{\partial}{\partial t_i}\phi_j(\mathbf{t})$$

for each $i = 2, 3, \dots$. *Then the pseudo-differential operator*

$$\mathcal{L} = K \circ \partial \circ K^{-1}$$

is a solution to the KP Hierarchy where K *is the unique, monic ordinary differential operator (with coefficients depending on* t_i *for* $i \geq 2$*) whose kernel is the* n*-dimensional space of functions span*$\langle \phi_1(\mathbf{t}), \dots, \phi_n(\mathbf{t}) \rangle$.

References

[1] D. J. Korteweg and G. de Vries, *XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **39** (1895), no. 240, 422-443, DOI 10.1080/14786449508620739, available at https://doi.org/10.1080/14786449508620739.

[2] D. G. Crighton, *Applications of KdV* (Michiel and Capel Hazewinkel Hans W. and de Jager, ed.), Springer Netherlands, Dordrecht, 1995.

[3] Alex Kasman N Benes K Young., *On Decompositions of the KdV 2-Soliton*, Journal of Nonlinear Science **16** (2006), no. 2, 179–200, DOI 10.1007/s00332-005-0709-2.

[4] Alex Kasman, *Glimpses of Soliton Theory*, Vol. 54, American Mathematical Society, 2010.

[5] A. V. Bäcklund, *Om ytor med konstant negativ krokning*, Lunds Univ. Arsskr **19** (1883), 1-41.

[6] W. R. Hamilton, *The Mathematical Papers of Sir William Rowan Hamilton*, Cambridge University Press, 2001.

[7] E. M. de Jager, *On the origin of the Korteweg-de Vries equation*, ArXiv Mathematics e-prints (2006), available at math/0602661.

[8] A. and Palacios M. Arribas and Elipe M., *Quaternions and the rotation of a rigid body*, Celestial Mechanics and Dynamical Astronomy **96** (2006), 239-251, DOI 10.1007/s10569-006-9037-6.

[9] L. D. Paniak, *Exact Noncommutative KP and KdV Multi-solitons*, ArXiv High Energy Physics - Theory e-prints (2001), available at hep-th/0105185.

[10] Alex Kasman, *On factoring an operator using elements of its kernel*, Communications in Algebra **45** (2017), no. 4, 1443-1451, DOI 10.1080/00927872.2016.1175612, available at https://doi.org/10.1080/00927872.2016.1175612.

[11] Israel Gelfand Pavel Etingof Vladmir Retakh, *Factorization of Differential Operators, Quasideterminants, and Nonabelian Toda Field Equations*, arXiv:q-alg/9701008v2 (1997), available at https://arxiv.org/abs/q-alg/9701008.

[12] Wen-Xiu Ma, *Complexiton solutions to integrable equations*, Nonlinear Analysis: Theory, Methods & Applications **63** (2005), no. 5, e2461 - e2471, DOI https://doi.org/10.1016/j.na.2005.01.068. Invited Talks from the Fourth World Congress of Nonlinear Analysts (WCNA 2004).

[13] Sen-Zhong Huang, *An operator method for finding exact solutions to vector Korteweg-de Vries equations*, Journal of Mathematical Physics **44** (2003), no. 3, 1357-1388, DOI 10.1063/1.1544414, available at https://doi.org/10.1063/1.1544414.

[14] L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, 2nd, WORLD SCIENTIFIC, 2003.

[15] Ryogo Hirota, *Structure of soliton equations* (Atsushi and Nimmo Nagai Jon and Gilson, ed.), Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2004.

[16] Mikio Sato, *Soliton Equations as Dynamical Systems on a Infinite Dimensional Grassmann Manifolds (Random Systems and Dynamical Systems)* **81** (1983).

[17] Chen Longxuan, *Definition of determinant and Cramer solutions over the quaternion field*, Acta Mathematica Sinica **7** (1991), no. 2, 171–180.

[18] Clifford S. and Greene Gardner John M. and Kruskal, *Method for Solving the Korteweg-deVries Equation*, Phys. Rev. Lett. **19** (1967), 1095–1097, DOI 10.1103/PhysRevLett.19.1095.

[19] N. J. and Kruskal Zabusky M. D., *Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States*, Phys. Rev. Lett. **15** (1965), 240–243, DOI 10.1103/PhysRevLett.15.240.

[20] I. M. and Retakh Gelfand V. S., *Determinants of matrices over noncommutative rings*, Functional Analysis and Its Applications **25** (1991), no. 2, 91–102, DOI 10.1007/BF01079588.

[21] Peter D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Communications on Pure and Applied Mathematics **21**, no. 5, 467-490, DOI 10.1002/cpa.3160210503, available at `https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160210503`.

[22] A. Zettl, *Factorization of differential operators* **27** (1971), 425–426.

[23] Boris Kupershmidt, *KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems*, Mathematical Surveys and Monographs, 2000.

[24] S. C. and Ngata Anco N. T. and Willoughby, *Interaction properties of complex modified Korteweg-de Vries (mKdV) solitons*, Physica D Nonlinear Phenomena **240** (2011), 1378-1394, DOI 10.1016/j.physd.2011.06.003, available at `1102.3620`.

[25] A. I. Arbab, *The Quaternionic Quantum Mechanics*, ArXiv e-prints (2010), available at `1003.0075`.

[26] S.I. Svinolupov, *Jordan Algebras and Generalized Korteweg-de Vries Equations*, Theoretical and Mathematical Physics **3** (1991), 611-620.

[27] Kit Ian and Liu Kou Wan-Kai and Xia, *Solve the linear quaternion-valued differential equations having multiple eigenvalues*, Journal of Mathematical Physics **60** (2019), no. 2, 023510, DOI 10.1063/1.5040237, available at `https://aip.scitation.org/doi/pdf/10.1063/1.5040237`.